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Pricing Power Options under the Heston Dynamics using the FFT

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Abstract: Numerous studies have presented evidence that certain financial assets may exhibit stochastic volatility or jumps, which cannot be captured within the Black-Scholes environment. This work investigates the valuation of power options when the variance follows the Heston model of stochastic volatility. A closed form representation of the characteristic function of the process is derived from the partial differential equation (PDE) of the replicating portfolio. The characteristic function is essential for the computation of the European power option prices via the Fast Fourier Transform (FFT) technique. Numerical results are presented.

Keywords: Power Option, Partial Differential Equation, Heston Model, Characteristic Function, Fast Fourier Transform

1. Introduction

Since Black & Scholes (1973) introduced the Black-Scholes model for option pricing, many scholars have tried to relax the assumptions made used in accordance to the model. This is because many studies have shown that in reality, certain financial assets may exhibit stochastic volatility or jumps. The evidence of this in option pricing has become an important issue because it gives possibility to model option pricing more accurately. One of the most accepted stochastic volatility models is due to Heston (1993). Such a model relaxes the constant volatility assumption made in the Black-Scholes approach. In this work, the asset price is assumed to follow the log-normal process governed by a single Brownian motion, with the volatility process driven by a second Brownian motion process. Both the asset price process and the volatility process are correlated by a constant correlation coefficient. With the assumption that the market is complete, a replicating portfolio technique is used in obtaining a partial differential equation (PDE). Consequently, using the PDE, the characteristic function of the logarithm of the underlying asset price is derived, which enables the application of the Fast Fourier Transform (FFT) for the computation of the power option prices. The FFT method has been used increasingly since it was first introduced in option pricing by Carr & Madan (1999). It is flexible approach in that it can encapsulate properties such as stochastic volatility, and still maintain its computational efficiency (see Pillay & O'Hara, 2011). Nevertheless, comparison between the FFT approach and Monte Carlo simulation (see Boyle, 1977) is demonstrated numerically to highlight the efficiency of the FFT technique.

2. The Model

Let $(\Omega, F, \mathbb{Q})$ be a probability space on which two Brownian motions, W_t^1 and W_t^2 for $t > 0$, are given. $F_t,\ 0 \leq t \leq T$ is the filtration generated by the Brownian motions and suppose $\mathbb{Q}$ is a risk-neutral probability. Given the underlying asset price S_t risk-free rate r and a constant factor β, Itô's Lemma implies that S_t^β is also a geometric Brownian motion following

$$dS_t^\beta = \left(\beta r + \frac{1}{2}\beta^2\sigma^2\right) S_t^\beta dt + \beta\sigma S_t^\beta dW_{t,1} \tag{2.1}$$

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We introduce an artificial asset $Z \equiv S_t^{\beta}$. Then Equation (2.1) becomes

$$dZ = \left(\beta r + \frac{1}{2}\beta^2\sigma^2 - \frac{1}{2}\beta\sigma^2\right)Zdt + \beta\sigma Z dW_{t,1},$$
$$dZ = (r-q)Zdt + \beta\sigma Z dW_{t,1}, \tag{2.2}$$

where $q = (1-\beta)\left(r + \frac{1}{2}\beta\sigma^2\right)$. From Equation (2.2), we observe the volatility is affected by a factor β. Hence, within the Heston environment, we propose the following model that governs the asset price process:

$$dZ = \left(\beta r + \frac{1}{2}\beta^2\sigma^2 - \frac{1}{2}\beta\sigma^2\right)Zdt + \beta\sigma Z dW_{t,1},$$
$$dv = \kappa(\theta - Y)dt + \sigma_0\sqrt{Y}dW_{t,2}, \tag{2.3}$$

where the variance, $Y = \beta^2 v$. Thus we can represent Equation (2.3) as follows:

$$dZ = \left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Zdt + \beta\sqrt{v}ZdW_{t,1}, \tag{2.4}$$

$$dv = \kappa(\theta - \beta^2 v)dt + \sigma_0\beta\sqrt{v}dW_{t,2}, \tag{2.5}$$

$$\langle dW_{t,1}dW_{t,2}\rangle = \rho dt, \tag{2.6}$$

where v follows a square-root mean reverting process, κ is the speed of the mean reversion, θ is the average level of the volatility, and ρ is the correlation coefficient between the two Brownian motions.

3. The Heston PDE for Power Options

Following Gatheral (2006), for a risk-neutral portfolio, we need to hedge the artificial asset and the random changes in the volatility. Assuming the market is complete (Esser, 2003), we consider a portfolio $\prod$ of an option with value *f*, $-\Delta$ units of Z and $-\phi$ units of another option with value g, to make the net amount equal to zero, which relies on the volatility,

$$\prod = f - \Delta Z - \phi g = 0 \tag{3.1}$$

Employing the two-dimensional extension of Itô's Lemma yields

$$df = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial f}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2} + \beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v \right.$$
$$\left. + \frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ\right]dt + \frac{\partial f}{\partial Z}\beta\sqrt{v}ZdW_{t,1} + \frac{\partial f}{\partial v}\sigma_0\beta\sqrt{v}dW_{t,2} \tag{3.2}$$

This is the same for dg that is:

$$dg = \left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2} + \beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v \right.$$
$$\left. + \frac{\partial^2 g}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ\right]dt + \frac{\partial g}{\partial Z}\beta\sqrt{v}ZdW_{t,1} + \frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}dW_{t,2} \tag{3.3}$$

The change in the portfolio $\prod$ in time dt is given by $d\prod = df - \Delta dZ - \phi dg$. It follows that by replacing the actual parameters yields,

$$
\begin{aligned}
d\prod = &\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial f}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v \right.\\
&\left. + \frac{1}{2}\frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right]dt + \frac{\partial f}{\partial Z}\beta\sqrt{v}Z dW_{t,1} + \frac{\partial f}{\partial v}\sigma_0\beta\sqrt{v}dW_{t,2}\\
&- \Delta\left[\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z dt + \beta\sqrt{v}Z dW_{t,1}\right]\\
&- \phi\left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 \right.\\
&\left. + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right]dt - \phi\left[\frac{\partial g}{\partial Z}\beta\sqrt{v}Z dW_{t,1} + \frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}dW_{t,2}\right]\\
&= \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial f}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 \right.\\
&\left. + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z - \Delta\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z\right]dt\\
&- \phi\left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 \right.\\
&\left. + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right]dt\left[\frac{\partial f}{\partial Z}\beta\sqrt{v}Z - \Delta\beta\sqrt{v}Z - \phi\frac{\partial g}{\partial Z}\beta\sqrt{v}Z\right]dW_{t,1}\\
&+ \left[\frac{\partial f}{\partial v}\sigma_0\beta\sqrt{v} - \phi\frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}\right]dW_{t,2}
\end{aligned}
\tag{3.4}
$$

Knowing that $\prod = 0$, we have $f = \Delta Z + \phi g$. In order to cancel out the randomness terms $dW_{t,1}$ and $dW_{t,2}$, we use the following:

$$
\frac{\partial f}{\partial Z} = \Delta + \phi\frac{\partial g}{\partial Z} \tag{3.5}
$$

$$
\frac{\partial f}{\partial v} = \phi\frac{\partial g}{\partial v}. \tag{3.6}
$$

For that reason,

$$
\begin{aligned}
d\prod = &\left[\frac{\partial f}{\partial t} + \left(\Delta + \phi\frac{\partial g}{\partial z}\right)\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \phi\frac{\partial g}{\partial z}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 \right.\\
&\left. + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z - \Delta\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z\right]dt\\
&- \phi\left[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial Z}\left(\beta r + \frac{1}{2}\beta^2 v - \frac{1}{2}\beta v\right)Z + \frac{\partial g}{\partial v}\kappa(\theta - \beta^2 v) + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 \right.\\
&\left. + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right]dt\\
&+ \left[(\Delta + \phi\frac{\partial g}{\partial Z})\beta\sqrt{v}Z - \Delta\beta\sqrt{v}Z - \phi\frac{\partial g}{\partial Z}\beta\sqrt{v}Z\right]dW_{t,1}\\
&+ \left[\phi\frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v} - \phi\frac{\partial g}{\partial v}\sigma_0\beta\sqrt{v}\right]dW_{t,2}\\
&= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 f}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right]dt\\
&- \phi\left[\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 v Z^2 + \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v + \frac{\partial^2 g}{\partial Z \partial v}\rho\sigma_0\beta^2 v Z\right]dt.
\end{aligned}
\tag{3.7}
$$

In order to avoid arbitrage opportunities, the portfolio should earn a risk-free rate r. Mathematically, this means

$$
d\prod = r\prod dt = r(f - \Delta Z - \phi g)dt.
$$

Since $\prod = 0$, then $d\prod = 0$. On that account, using the respective Equation (3.5), Equation (3.6) and Equation (3.7) renders

$$\frac{\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ-rf+rZ\frac{\partial f}{\partial Z}}{\frac{\partial f}{\partial v}} = \frac{\frac{\partial g}{\partial t}+\frac{1}{2}\frac{\partial^2 g}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 g}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 g}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ-rg+rZ\frac{\partial g}{\partial Z}}{\frac{\partial g}{\partial v}} \tag{3.8}$$

Following Heston (1993), both sides of Equation (3.8) are equal to some function, say h such that $h(Z,v)=-\kappa(\theta-\beta^2 v)+\lambda(Z,v)$, where $\lambda(Z,v)=\lambda\beta^2 v$ is the volatility risk premium. Thence,

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ-rf+rZ\frac{\partial f}{\partial Z}=\frac{\partial f}{\partial v}[-\kappa(\theta-\beta^2 v)+\lambda\beta^2 v]. \tag{3.9}$$

Suppose we have the risk-neutral measures be $\kappa^*=\kappa+\lambda$, and $\theta^*=\frac{\kappa\theta}{\kappa+\lambda}$. This cancels out the volatility risk premium. Consequently, the stochastic process followed by the variance is now

$$dv=\kappa^*(\theta^*-\beta^2 v)dt+\sigma_0\beta\sqrt{v}dW_{t,2}. \tag{3.10}$$

It follows that Equation (3.9) becomes

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial Z^2}\beta^2 vZ^2+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial Z\partial v}\rho\sigma_0\beta^2 vZ+\frac{\partial f}{\partial v}\kappa^*(\theta^*-\beta^2 v)+rZ\frac{\partial f}{\partial Z}-rf=0, \tag{3.11}$$

which has a similar form to the Heston PDE. Assume that $X\equiv lnZ$. Solving for its partial derivatives, and then substituting the results into Equation (3.9) returns

$$0=\frac{\partial f}{\partial t}+\left(r-\frac{1}{2}\beta^2 v\right)\frac{\partial f}{\partial X}+\frac{1}{2}\frac{\partial^2 f}{\partial X^2}\beta^2 v+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma_0^2\beta^2 v+\frac{\partial^2 f}{\partial X\partial v}\rho\sigma_2\beta^2 v+\frac{\partial f}{\partial v}\kappa^*(\theta^*-\beta^2 v)-rf. \tag{3.12}$$

4. Deriving the Characteristic Function

So as to solve for the characteristic function, we conjecture that the solution for the PDE (3.12) has the following form:

$$\begin{aligned} f &= Ze^{(\beta-1)\left(r+\frac{1}{2}\beta v\right)\tau}P_1-Ke^{-r\tau}P_2 \\ &= e^{X+(\beta-1)\left(r+\frac{1}{2}\beta v\right)\tau}P_1-Ke^{-r\tau}P_2. \end{aligned} \tag{4.1}$$

It follows that by replacing the form (4.1) into the PDE (3.12) yields

$$\begin{aligned} 0 = e^{X+(\beta-1)(r+\frac{1}{2}\beta v)\tau}\Bigg\{ & -(\beta-1)\left(r+\frac{1}{2}\beta v\right)P_1 \\ & +\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\left[\rho\sigma_0\beta^2 v+\kappa^*(\theta^*-\beta^2 v)+\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\frac{1}{2}\sigma_0^2\beta^2 v\right]P_1 \\ & +\left[r+\frac{1}{2}\beta^2 v+\rho\sigma_0\beta^2 v\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\right]\frac{\partial P_1}{\partial X}+\frac{1}{2}\frac{\partial^2 P_1}{\partial X^2}\beta^2 v \\ & +\left[\rho\sigma_0\beta^2 v+\kappa^*(\theta^*-\beta^2 v)+\sigma_0^2\beta^2 v\left(\frac{1}{2}\beta^2\tau-\frac{1}{2}\beta\tau\right)\right]\frac{\partial P_1}{\partial v}+\frac{1}{2}\frac{\partial^2 P_1}{\partial v^2}\sigma_0^2\beta^2 v \\ & +\frac{\partial^2 P_1}{\partial X\partial v}\rho\sigma_0\beta^2 v-\frac{\partial P_1}{\partial \tau}\Bigg\} \\ & -Ke^{-r\tau}\Bigg[\left(r-\frac{1}{2}\beta^2 v\right)\frac{\partial P_2}{\partial X}+\frac{1}{2}\frac{\partial^2 P_2}{\partial X^2}\beta^2 v+\frac{\partial P_2}{\partial v}\kappa^*(\theta^*-\beta^2 v)+\frac{1}{2}\frac{\partial^2 P_2}{\partial v^2}\sigma_0^2\beta^2 v \\ & +\frac{\partial^2 P_2}{\partial X\partial v}\rho\sigma_0\beta^2 v-\frac{\partial P_2}{\partial \tau}\Bigg]. \end{aligned} \tag{4.2}$$

This can be represented as $e^{X+(\beta-1)(r+\frac{1}{2}\beta v)\tau}[f(P_1)]-Ke^{-r\tau}[f(P_2)]=0$, where one possible solution is $f(P_1)=f(P_2)=0$. Alternatively, this implies that P_1 and P_2 must satisfy the PDEs

$$0 = \left[-(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + \frac{1}{2}\sigma_0^2\beta^2 v c_j^2 + \left(a - b_j\beta^2 v\right)\mathrm{c}_j\right]P_j + \left(r + l_j\beta^2 v + \rho\sigma_0\beta^2 v \mathrm{c}_j\right)\frac{\partial P_j}{\partial X}$$
$$+ \frac{1}{2}\frac{\partial^2 P_j}{\partial X^2}\beta^2 v + \frac{1}{2}\frac{\partial^2 P_j}{\partial v^2}\sigma_0^2\beta^2 v + \left[\left(a - b_j\beta^2 v\right) + \sigma_0^2\beta^2 v \mathrm{c}_j\right]\frac{\partial P_j}{\partial v} + \frac{\partial^2 P_j}{\partial X \partial v}\rho\sigma_0\beta^2 v \tag{4.3}$$
$$-\frac{\partial P_j}{\partial \tau},$$

for $j = 1,2$, where $k_1 = 1, k_2 = 0, c_1 = \frac{1}{2}\beta^2\tau - \frac{1}{2}\beta\tau, c_2 = 0,\ l_1 = \frac{1}{2},\ l_2 = -\frac{1}{2}, b_1 = \kappa^* - \rho\sigma_0, b_2 = \kappa^*,$ and $a = \kappa^*\theta^*$. We now investigate the characteristic function within the Heston framework for power options. We suggest that the characteristic function has the following form:

$$\varphi_j(X, v, t; u) = \exp\left(C_j + D_j\beta^2 v + iuX\right). \tag{4.4}$$

Accordingly, we substitute the characteristic function (4.4) into PDE (4.3),

$$0 = \left[-(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + \frac{1}{2}\sigma_0^2\beta^2 v c_j^2 + (a - b_j\beta^2 v)c_j\right]\varphi_j + \left(r + l_j\beta^2 v + \rho\sigma_0\beta^2 v c_j\right)iu\varphi_j$$
$$+ \frac{1}{2}\left(-u^2\varphi_j\right)\beta^2 v + \frac{1}{2}\left(\beta^4 D_j^2\varphi_j\right)\sigma_0\beta^2 v + \left[\left(a - b_j\beta^2 v\right) + \sigma_0^2\beta^2 v c_j\right]B^2 D_j\varphi_j$$
$$+ \rho\sigma_0\beta^2 v\left(\beta^2 D_j iu\varphi_j\right) - \frac{\partial C_j}{\partial \tau}\varphi_j - \beta^2 v\frac{\partial D_j}{\partial \tau}\varphi_j \tag{4.5}$$
$$= \beta^2 v\left[\frac{1}{2}\sigma_0^2 c_j^2 - b_j c_j + l_j iu + \rho\sigma_0 c_j iu - \frac{1}{2}u^2 + \frac{1}{2}\sigma_0^2\beta^4 D_j^2 - b_j\beta^2 D_j + \sigma_0^2 c_j\beta^2 D_j\right.$$
$$\left. + \rho\sigma_0 iu\beta^2 D_j - \frac{\partial D_j}{\partial \tau}\right] + \left[-(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + ac_j + riu + a\beta^2 D_j - \frac{\partial C_j}{\partial \tau}\right]$$

subject to the following boundary conditions: $C_j(0) = 0$, and $D_j(0) = 0$. This reduces to solving two ordinary differential equations (ODE),

$$\frac{\partial D_j}{\partial \tau} = \left(\frac{1}{2}\sigma_0^2 c_j^2 - b_j c_j + l_j iu + \rho\sigma_0 c_j iu - \frac{1}{2}u^2\right) + \left(\sigma_0^2 c_j\beta^2 + \rho\sigma_0 iu\beta^2 - \beta^2 b_j\right)D_j + \frac{1}{2}\sigma_0^2\beta^4 D_j^2, \tag{4.6}$$

$$\frac{\partial C_j}{\partial \tau} = -(\beta-1)\left(r+\frac{1}{2}\beta v\right)k_j + ac_j + riu + a\beta^2 D_j. \tag{4.7}$$

Equation (4.6) is nonlinear and is of the form of a Riccati equation. Any equation of the Riccati type can always be transformed to the following second order linear homogeneous ordinary differential equation (Bastami et al., 2010) using a substitution $D_j = -\frac{Y'}{YR}$:

$$Y'' - \left(Q + \frac{R'}{R}\right)Y' + PRY = 0, \tag{4.8}$$

where $P = \frac{1}{2}\sigma_0^2 c_j^2 - b_j c_j + l_j iu + \rho\sigma_0 c_j iu - \frac{1}{2}u^2, Q = \sigma_0^2 c_j\beta^2 + \rho\sigma_0 iu\beta^2 - \beta^2 b_j$ and $R = \frac{1}{2}\sigma_0^2\beta^4$. Making further substitutions, $a = 1, b = -(Q + \frac{R'}{R})$ and $c = PR$, the ODE (4.8) is now,

$$\begin{aligned} &a\frac{\partial^2 Y}{\partial \tau^2} + b\frac{\partial Y}{\partial \tau} + cY = 0, \\ &aY'' + bY' + cY = 0. \end{aligned} \tag{4.9}$$

Besides, the characteristic equation of the ODE (4.9) is $ar^2 + br + c = 0$ which is a quadratic equation with roots,

$$r_{1,2} = \frac{\beta^2\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right) \pm \beta^2\sqrt{\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right)^2 - \sigma_0^2\left(\sigma_0^2 c_j^2 - 2b_j c_j + 2l_j iu + 2\rho\sigma_0 c_j iu - u^2\right)}}{2}. \tag{4.10}$$

Suppose that r_1 and r_2 are distinct real numbers, then the general solution is of the form, $Y = A_1 e^{r_1\tau} + A_2 e^{r_2\tau}$, where $Y' = \frac{\partial Y}{\partial \tau} = r_1 A_1 e^{r_1\tau} + r_2 A_2 e^{r_2\tau}$. Replacing this back into $D_j = -\frac{Y'}{YR}$ yields,

$$D_j = \frac{-r_1 A_1 e^{r_1\tau} - r_2 A_2 e^{r_2\tau}}{R(A_1 e^{r_1\tau} + A_2 e^{r_2\tau})}. \tag{4.11}$$

Recall the terminal condition $D(0) = 0$. It follows that

$$-\frac{r_1}{r_2} = \frac{A_2}{A_1}.$$

Carrying on with the calculation,

$$D_j = \frac{-\left\{e^{r_1\tau}\left[r_1 c_1 + r_2\left(-\frac{c_1 r_1}{r_2}\right)\frac{e^{r2\tau}}{e^{r_1\tau}}\right]\right\}}{R\left\{e^{r_1\tau}\left[c_1 + \left(-\frac{c_1 r_1}{r_2}\right)\frac{e^{r2\tau}}{e^{r_1\tau}}\right]\right\}} = -\frac{1}{R}\left[\frac{r_1 - r_1(e^{r_2\tau})(e^{-r_1\tau})}{1 - \frac{r_1}{r_2}(e^{r_2\tau})(e^{-r_1\tau})}\right] = -\frac{r_1}{R}\left[\frac{1 - e^{(r_2 - r_1)\tau}}{1 - \frac{r_1}{r_2}e^{(r_2 - r_1)\tau}}\right]. \tag{4.12}$$

We now define the following,

$$d = r_1 - r_2 = \beta^2\sqrt{(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j)^2 - \sigma_0^2(\sigma_0^2 c_j^2 - 2b_j c_j + 2l_j iu + 2\rho\sigma_0 c_j iu - u^2)},$$

$$g = \frac{r_1}{r_2} = \frac{\beta^2\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right) + d}{\beta^2\left(\sigma_0^2 c_j + \rho\sigma_0 iu - b_j\right) - d}.$$

Thus, continuing to solve (4.12),

$$D_j = \frac{\beta^2\left(b_j - \rho\sigma_0 iu - \sigma_0^2 c_j\right) - d}{\sigma_0^2\beta^4}\left(\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}\right). \tag{4.13}$$

Given the solution in (4.13), we can now solve the ODE (4.7) by first integrating both sides of the ODE.

$$C_j = \left[-(\beta - 1)\left(r + \frac{1}{2}\beta v\right)k_j + ac_j + riu\right]\tau + \frac{a}{\sigma_0^2\beta^2}\left[\left[\beta^2\left(b_j - \rho\sigma_0 iu - \sigma_0^2 c_j\right) - d\right]\tau - 2ln\left(\frac{1 - ge^{-d\tau}}{1 - g}\right)\right]. \tag{4.14}$$

We have now obtained solutions for the ODE as given by (4.6) and (4.7), which are shown in (4.13) and (4.14), respectively. Choosing $j = 2$, and replacing the solutions into Equation (4.4) results to the following,

$$\begin{aligned} \varphi\beta &= \exp(iuX) \\ &\times exp\left\{riu\tau + \frac{\kappa^*\theta^*}{\sigma_0^2\beta^2}\left[\left[\beta^2(\kappa^* - \rho\sigma_0 iu) - d\right]\tau - 2ln\left(\frac{1 - ge^{-d\tau}}{1 - g}\right)\right]\right\} \\ &\times exp\left\{\frac{v}{\sigma_0^2\beta^2}\left[\beta^2(\kappa^* - \rho\sigma_0 iu) - d\left(\frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}\right)\right]\right\}, \end{aligned} \tag{4.15}$$

where

$$d = \beta^2\sqrt{(\rho\sigma_0 iu - \kappa^*)^2 + \sigma_0^2(iu + u^2)},$$

$$g = \frac{\beta^2(\rho\sigma_0 iu - \kappa^*) + d}{\beta^2(\rho\sigma_0 iu - \kappa^*) + d}.$$

Using the result in (4.15), we can now apply the Fast Fourier Transform technique to price the power option when the volatility is stochastic.

5. Power Option Pricing using the Fast Fourier Transform

The essence behind the FFT approach is the characteristic function of the stochastic process. Provided that this is obtained analytically, we can use this approach to price the options. The characteristic function is defined as follows:

Definition 5.1: (Characteristic Function). *For a one-dimensional stochastic process $X_t, 0 \leq t \leq T$, the characteristic function is the Fourier transform of the probability density function $q_T(X_T)$ given as follows:*

$$\varphi(v) = \mathbb{E}_{\mathbb{Q}}(e^{ivX_T}) = \int_{-\infty}^{\infty} e^{ivX_T} q_T(X_T) dX_T. \tag{5.1}$$

Let K be the strike price and T the maturity of a power option with terminal asset price S_T^{β}, which is governed by the dynamics (2.1). The price of a power call option is computed as the discounted risk-neutral conditional expectation of the terminal payoff $(S_T^{\beta} - K)^+ = \max(S_T^{\beta} - K, 0)$:

$$PC(S_T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S_T^{\beta} - K\right)^+ |\mathcal{F}_{\mathrm{t}}\right], \tag{5.2}$$

where r is a constant interest rate. We define $X_t = lnS_t$ and $k = \frac{lnK}{\beta}$. Moreover, we express the option pricing function (5.2) as a function of the log strike k instead of the terminal log asset price X_T,

$$PC_T(k)e^{-rT}\int_{k}^{\infty}(e^{X_T} - e^k)f_T(X_T)d(X_T), \tag{5.3}$$

where $f_T(X_T)$ is the density function of the process X_T. Following Carr & Madan (1999), for $\alpha > 0$, we define a modified power call price,

$$\hat{P}C_T(k) = e^{\alpha k}PC_T(k), \tag{5.4}$$

where the Fourier Transform (FT) of $\hat{P}C_T(k)$ is given by:

$$\mathcal{F}_{\beta}(v) = \int_{-\infty}^{\infty} e^{ivk}\hat{P}C_T(k)dk. \tag{5.5}$$

Applying the inverse FT to (5.5), then substituting (5.4) with (5.3) into (5.5), and also by the definition of the characteristic function (5.1), we obtain the price of a power call option as follows:

$$PC_T(k) = \frac{e^{-\alpha k}}{2\pi}\left[\int_{-\infty}^{\infty} e^{ivk}e^{-rT}\frac{\varphi_{\beta}[v - i(\alpha+1)]}{(\alpha+iv)(\alpha+iv+1)}dv\right], \tag{5.6}$$

where

$$\mathcal{F}_{\beta}(v) = e^{-r\tau}\frac{\varphi_{\beta}[v - i(\alpha+1)]}{(\alpha+iv)(\alpha+iv+1)}. \tag{5.7}$$

Thus for an efficient implementation of the FFT, a closed-form representation of the characteristic function $\varphi_{\beta}(v)$ is needed, which we have shown earlier, has the form of (4.15). Given the pricing function (5.6), we can price the power call option as follows:

$$PC_T(k_u) = \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)}e^{ibv_j}\,\mathcal{F}_{\beta}(v_j)\frac{\eta}{3}\left[3 + (-1)^j - \delta_{j-1}\right], \tag{5.8}$$

where $v_j = \eta(j-1), k_u = -b + \omega(u-1), b = \frac{N\omega}{2}, \omega\eta = \frac{2\pi}{N}$, and δ_n is the Kronecker delta function which is unity for $n = 0$ and zero otherwise. The choice of ω and η is essential because it governs this approach. A small ω gives us a range of prices across a wide range of strike prices; while a large value of η can give inaccurate prices. Moreover, the FFT is an algorithm that evaluate the summations of the following form efficiently:

$$X(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)}x(j), \quad k = 1, \dots, N \tag{5.9}$$

with $x(j) = e^{ibv_j}\,\mathcal{F}_{\beta}(v_j)\frac{\eta}{3}\left[3 + (-1)^j - \delta_{j-1}\right]$. Hence, the presentation of the power call price in the form (5.8) is a special case of (5.9) which enables the use of the FFT.

6. Power Option Pricing using Monte Carlo Simulation

Consider the problem of pricing a power call option of the form (5.2), as exhibited in Section 5. For application of the Monte Carlo simulation, we apply the fully truncated Euler scheme by Lord et al. (2010).

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which is defined two standard Wiener processes W_t^S with respect to the underlying, and W_t^v, with respect to the volatility. Let $\mathcal{F}_t, 0 \leq t \leq T$, be the filtration generated by these Brownian motion. Suppose $\mathbb{Q}$ is a risk-neutral probability under which the asset price process $S_t, 0 \leq t \leq T$ is governed by dynamics given in (2.4) and (2.5). To facilitate the discretization, we consider the log-asset price $X_t = lnS_t^{\beta}$. Applying Itô's Lemma to this function yields the following log-asset price Dynamics

$$dX_t = \beta\left(r - \frac{1}{2}v\right)dt + \beta\sqrt{v}dW_t^S. \tag{6.1}$$

Suppose we approximate the paths of the log asset price process (6.1) and the stochastic volatility process (2.5), on a discrete time grid via Euler discretization. Let $[t = t_0 < t_1 < \cdots < t_M = T]$ be a partition of the time interval $[t, T]$ into M equal segments of length Δt_i that is $t_i = \frac{iT}{M}$ for each $i = 0,1, \dots, M$. The fully truncated Euler discretization of the log asset price process is

$$\hat{X}_i = \hat{X}_{i-1} + \beta\left(r - \frac{1}{2}\hat{v}_t^+\right)\Delta t_i + \beta\sqrt{\hat{v}_{i-1}^+ \Delta t_i}Z_S, \tag{6.2}$$

$$\hat{v}_i = \hat{v}_{i-1}^+ + \kappa(\theta - \beta^2\hat{v}_{i-1}^+)\Delta t_i + \beta\sigma\sqrt{\hat{v}_{i-1}^+ \Delta t_i}Z_v, \tag{6.3}$$

where $\hat{v}_i^+ = \max(\hat{v}_i, 0)$, $Z_v \sim N(0,1)$ and $Z_S = \rho Z_v + \sqrt{1-\rho^2}Z$, where $Z \sim N(0,1)$. Using the Milstein scheme, the discretization of the volatility process (6.3) is:

$$\hat{v}_i = \hat{v}_{i-1}^+ + \kappa(\theta - \beta^2\hat{v}_{i-1}^+)\Delta t_i + \beta\sigma\sqrt{\hat{v}_{i-1}^+ \Delta t_i}Z_v + \frac{1}{4}\sigma^2\Delta t_i(Z_v^2 - 1). \tag{6.4}$$

We simulate the diffusion part of the log asset price by drawing a random sample from a normal distribution with mean 0 and standard deviation 1 for both Z_S and Z_v for each $i = 0,1, \dots, M$, and obtain a log asset price for the maturity date of the option, $\hat{X}_M = \hat{X}_T$. By repeating this procedure, many paths can be generated. The price of a power call option (5.2) can be estimated by Monte Carlo simulation using

$$PC(t, \hat{X}_T) = \frac{e^{-r(T-t)}}{n}\sum_{i=1}^{n} \max\left(e^{\hat{X}_T^i} - K, 0\right), \tag{6.5}$$

where n is the number of sample paths used in simulation and $\hat{X}_T$ denotes the simulated value of X_t over each sample path using M time steps. This Monte Carlo estimator converges to the correct price $PC(t, S_T)$ as the number of time steps M and the number of samples n become large.

7. Numerical Results

In this section, we present a numerical comparison between the Fast Fourier Transform (FFT) approach and the Monte Carlo simulation technique. We apply the two approaches for the pricing of a power call option with stochastic volatility with a view to comparing the performance of the two techniques.[1]

We employed the FFT scheme with $N = 2^{10}, \delta = 1.25$, and α between [0.28, 0.32] to minimize the relative error between the results obtained from both techniques. Linear interpolation is applied to obtain a single option price corresponding to the respective strike price. For the Monte Carlo simulation, we employed the Milstein scheme because this produce better result than the Euler discretization. We take N = 500, 000 sample paths, and partition the time interval $[0, T]$ into $m = 200$ equal segments. Since we are only considering the comparison of the accuracy and efficiency of the models, we do not calibrate the model parameters, and rather we use the following hypothetical parameters of $S = 2, r = 0.08, T = 1, \kappa = 2, \theta = 0.04, \sigma_0 = 0.1, \rho = -0.5$ and $v_0 = 0.05$.

[1] The codes were written in MATLAB, and the computations were conducted on an Intel Core 2 Duo processor P8400 @2.26 GHz machine running under Windows Vista Service Pack 2 with 2 GB RAM.

Table 1 compares the pricing accuracy between the two techniques across a range of strike prices, as well as the relative error (in percentage) between the two prices. Using the Monte Carlo simulation as the benchmark, it demonstrates the efficiency of the FFT technique over the Monte Carlo simulation technique.

Strike, K	FFT	Computation Time (seconds)	Monte Carlo	Computation Time (seconds)	% Difference
0.5	3.9480	0.003092	3.9346	26.196885	0.340568292
1.0	3.4626	0.003568	3.4742	27.867434	0.333889816
1.5	3.0064	0.003087	3.0130	28.040573	0.219050780
2.0	2.5508	0.003033	2.5526	24.529332	0.070516336
2.5	2.1001	0.003479	2.0904	30.537946	0.464026024
3.0	1.6456	0.003223	1.6427	24.736727	0.176538625
3.5	1.2163	0.003309	1.2196	24.399634	0.270580518
4.0	0.8450	0.003407	0.8476	24.733935	0.306748466

Table1: Comparison of prices for the power call option with stochastic volatility computed by FFT and Monte Carlo simulation

8. Conclusion

In this paper, we provide a valuation of power options under the Heston dynamics using the fast Fourier transform (FFT) technique. We present an analytical form of the characteristic function which is derived from the partial differential equation (PDE) of the replicating portfolio. The numerical results show that the FFT technique is more efficient than the Monte Carlo simulation.

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Image Encryption Using Stream Cipher Based on Nonlinear Combination Generator with Enhanced Security

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Abstract: The images are very largely used in our daily life; the security of their transfer became necessary. In this work a novel image encryption scheme using stream cipher algorithm based on nonlinear combination generator is developed. The main contribution of this work is to enhance the security of encrypted image. The proposed scheme is based on the use the several linear feedback shifts registers whose feedback polynomials are primitive and of degrees are all pairwise coprimes combined by resilient function whose resiliency order, algebraic degree and nonlinearity attain Siegenthaler's and Sarkar, al.'s bounds. This proposed scheme is simple and highly efficient. In order to evaluate performance, the proposed algorithm was measured through a series of tests. These tests included visual test and histogram analysis, key space analysis, correlation coefficient analysis, image entropy, key sensitivity analysis, noise analysis, Berlekamp-Massey attack, correlation attack and algebraic attack. Experimental results demonstrate the proposed system is highly key sensitive, highly resistance to the noises and shows a good resistance against brute-force, statistical attacks, Berlekamp-Massey attack, correlation attack, algebraic attack and a robust system which makes it a potential candidate for encryption of image.

Keywords: Cryptosystem, Decryption, Image Correlation, Image Encryption, Key Stream, Nonlinear combination Generator, Resilient Function.

1. Introduction

The numerical networks knew a strong growth in the last few years. The majority of these networks are inter-connected and connected to Internet which is considered today as a motorway where circulates freely a quantity of information increasingly important. The transmitted information is not exclusively in the form of textual but also audio data, digital images and other multi-media. The circulation of the images on these networks is very largely used in our daily life, and more their use is increasing, more their safety is vital. For example, the images to be transmitted can be collected and copied during their course without losses of quality. The intercepted images can be thereafter the subject of an exchange of information and illegal numerical storage. It is thus necessary to make incomprehensible of the transferred files and to protect them from any undesirable interception. The modern cipher of the data is very often the only effective means to answer these requirements.

In this paper, we are interested in the security of the data images, which are regarded as particular data because of their sizes and their information which is two-dimensional and redundant natures. These characteristics of the data make the classical cryptographic algorithms such as DES, RSA, and ... are inefficient for image encryption due to image inherent features, especially high volume image data. Many researchers proposed different image encryption schemes to overcome image encryption problems [1, 2, 3, 4]. In this approach we have tried to find a simple, fast and secure algorithm for image encryption using a stream cipher algorithm based on the combination of several linear feedback shift registers (LFSRs) by a Boolean function satisfying all the criteria cryptographic necessary to carry out a maximum safety. Finally, this algorithm is robust and very sensitive to small changes in key so even with the knowledge of the key approximate values; there is no possibility for the attacker to break the cipher.

2. Stream Cipher Based on Nonlinear combination Generator

Most practical stream ciphers are based on linear feedback shift registers (LFSRs). An LFSR of length L is a L stage register with a linear feedback function. During its operation contents of each storage unit are shifted to the next unit and the output of the feedback function is fed to the last storage unit. If the feedback function of the LFSR is primitive and its initial state is a non-zero state, then the output sequence produced by the LFSR has the maximum period of $2^L - 1$.

A system of stream cipher based on nonlinear combination generator generally breaks up into three parts: An engine, primarily made up of linear feedback shift register with maximum period, the goal of this engine is to provide one or more continuations, having good statistical properties already; generally, registers (LFSRs) are used whose feedback polynomials are primitives and of degrees are all pairwise coprimes.

The outputs of the (LFSRs), having more or less strong properties of linearity, it is essential to make disappear to the maximum these properties of linearity. The second part is thus a module whose role is to break this linearity by combining the outputs to (LFSRs) by a nonlinear Boolean function having the best possible cryptographic properties. These a nonlinear Boolean function must be selected very carefully to offer a resistance to the attacks.

A module of combination the key stream with the plaintext, most common is reduced to a modulo2 addition (XOR). The key stream $(z_t)_{t\geq 0}$ is generated as a nonlinear function f of the outputs of the component LFSRs. Such key stream generator is called nonlinear combination generator, and f is called the combining function. The outputs of f is bitwise XORed with the plaintext $(m_t)_{t\geq 0}$ to produce the cipher text $(c_t)_{t\geq 0}$, this construction is illustrated in figure 1.

The combining function must have high algebraic degree, high nonlinearity and good correlation immunity to prevent correlation and linear attacks [5, 6, 7, 8]. It must also have high algebraic immunity to provide resistance against the algebraic attacks. [9, 10, 11, 12, 13, 14].

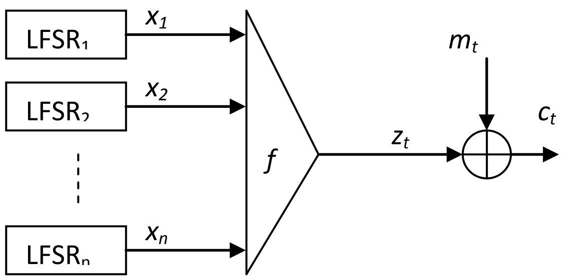

Fig. 1. System of Stream Cipher Based on Nonlinear Combination Generator;

3. Proposed Approach

We used stream ciphers algorithm based on nonlinear combination generator for constructing our new approach. The layout of our method is presented in Figure 2. Let *13* LFSRs denoted $R_1, R_2, ..., R_{13}$ whose respectively length $L_1, L_2, ..., L_{13}$ are pairwise distinct greater than 2, are combined by a nonlinear function f as in figure 1 which is expressed in algebraic normal form. Denote the output of R_i at time t by $s_i(t)$. Then the key stream $z(t)$ is given as

$$z(t) = f\left(s_1(t), s_2(t), ..., s_{13}(t)\right) \tag{1}$$

The linear complexity of the key stream is $\lambda(s) = f\left(L_1, L_2, ..., L_{13}\right)$ is evaluated over the integers rather than over Z_2.

Let Y an original image (plain-image) of $n \times m$ pixels. First, sender transforms the plain image Y into binary array (plain image digit). Let $y(t), c(t)$ and $z(t)$ be the plain image digit, cipher image digit and key stream digit at time t. Then the encryption process can be described by the equation

$$c(t) = y(t) \oplus z(t) \tag{2}$$

where $\oplus$ is the function XOR (Or exclusive).

The cipher image digit $c(t)$ is sent to the receiver over an unsecure channel and is decrypted a bitwise XOR operation the key stream digit and the plaint image digit can be described as

$$y(t) = c(t) \oplus z(t) \tag{3}$$

The cipher image digit at the receiver is decrypted by producing the same key stream. The receiver transforms the decrypt image digit in to plain image y of $n \times m$ pixels.

Their main advantages are their extreme speeds and their capacity to change every symbol of the plaintext. Besides, they are thus used in a privileged way in the case of communications likely to be strongly disturbed because they have the advantage of not propagating the errors [15].

3.1. Key K

The secret key K of the cryptosystem is then either made up of the initialization of only one register but of 13 registers is a chain of bits length $53+59+61+67+71+73+79+83+89+91+95+101+102=1024$ bits. This chain of bits must be sufficiently large in order to guarantee a maximum security and also to avoid, at the present time and with reasonable means, any attempt at against brute-force attack.

3.2. LFSRs

We considered thirteen maximum-length LFSRs whose lengths L_i, $i \in [1,...,13]$ are all pairwise coprimes which feedback polynomials are respectively $p_1,..., p_{13}$. We chose the following feedback polynomials:

$p_1(x) = x^{53} + x^6 + x^2 + x + 1$, $p_2(x) = x^{59} + x^{22} + x^{21} + x + 1$,

$p_3(x) = x^{61} + x^5 + x^2 + x + 1$, $p_4(x) = x^{67} + x^5 + x^2 + x + 1$,

$p_5(x) = x^{71} + x^5 + x^3 + x + 1$, $p_6(x) = x^{73} + x^4 + x^3 + x^2 + 1$,

$p_7(x) = x^{79} + x^4 + x^3 + x^2 + 1$, $p_8(x) = x^{83} + x^7 + x^4 + x^2 + 1$,

$p_9(x) = x^{89} + x^6 + x^5 + x^3 + 1$, $p_{10}(x) = x^{91} + x^7 + x^6 + x^5 + x^3 + x^2 + 1$,

$p_{11}(x) = x^{95} + x^6 + x^5 + x^4 + x^2 + x + 1$, $p_{12}(x) = x^{101} + x^7 + x^6 + x + 1$

$p_{13}(x) = x^{102} + x^6 + x^5 + x^3 + 1$.

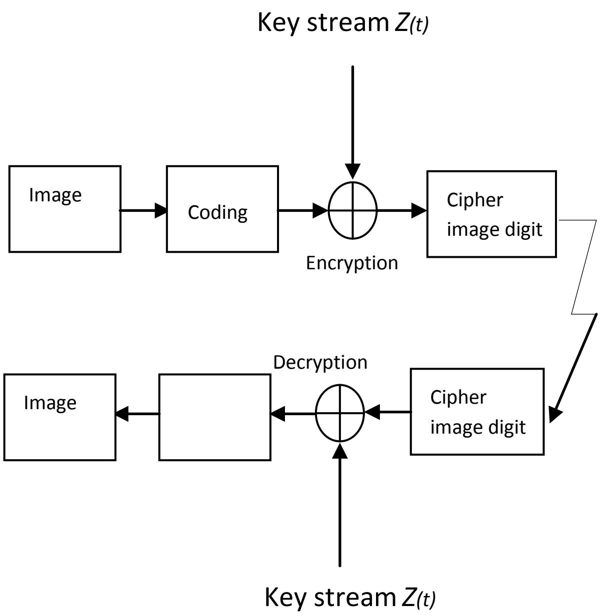

Fig. 2. Block Diagram of the Proposed Approach;

3.3. Nonlinear Combining Function

The combining function used for generating the key stream is a Boolean function f from F_2^{13} into F_2. At each time t, thirteen sequences bits $s_1(t), s_2(t), ..., s_{13}(t)$ are inputs to the Boolean function f to calculate the key stream $z(t)$ as show equation (1).

The combining function f used in our approach is presented in [16]. This function is 5-resilient function, of algebraic degree 7 and nonlinearity $Nf = 3969$ with algebraic immunity 6, satisfies all the cryptographic criteria necessary carrying out the best possible compromises.

3.4. Algorithm1: Encryption and Decryption Image Algorithm

Encryption

1. *Load the plain-image Y (i.e. Original image);*
2. *Transform the plain-image into column digit (i.e. plain image digit) and to store them in y;*
3. $N \leftarrow$ *the length of y;*
4. *for $t = 1$ to N to make ;*
5. *To generate the key-stream $z(t) = f\left(s_1(t), s_2(t), ..., s_n(t)\right)$ as show the algorithm 2 ;*
6. *End to make ;*
7. *for $t = 1$to N to make*
8. *Calculate the cipher image digit using relation $c(t) = XOR\left(y(t), z(t)\right)$;*
9. *End to make ;*
10. *Sent the cipher image digit.*

Decryption

1. *Load the cipher-image digit c*
2. $N \leftarrow$ *the length of c;*
3. *for $i = 1$ to N to make ;*
4. *To generate the key stream $z(t) = f\left(s_1(t), s_2(t), ..., s_{13}(t)\right)$ as show the algorithm 2;*

5. *End to make ;*
6. *for* $t = 1$ *to* N *to make;*
7. *Calculate the decipher image digit using relation* $y(t) = xor\left(c(t), z(t)\right)$ *;*
8. *End to make ;*
9. *To put the decipher image digit* y *in the form of an image of* $n \times m$ *pixels and to store it in* Y *;*

3.5. Algorithm2: Key stream

1. *To read* N *, length of* y *;*
2. *To introduce the secret key, the value of initialization of 13 registers ;*
3. *for* $t = 1$ *to* N *to make;*
4. *To generate the output of* $s_1(t), s_2(t), \ldots, s_{13}(t)$ *;*
5. *End to make ;*
6. *for* $t = 1$ *to* N *to make;*
7. *To generate the key stream* $z(t) = f\left(s_1(t), s_2(t), \ldots, s_{13}(t)\right)$ *;*
8. *End to make.*

4. Test Results

In this section, the performance of the proposed image encryption scheme is analyzed in detail. We discuss the security analysis of the proposed image encryption scheme including some important ones like statistical sensitivity, key sensitivity analysis, key space analysis etc. to prove the proposed cryptosystem is secure against the most common attacks.

4.1. Visual Testing

A number of images are encrypted and decrypted by the proposed method, and visual test is performed. Two examples are shown in Fig. 3 (a) and Fig. 3 (d), with respectively 128 x 128 and 256x256 pixels. By comparing the original and the encrypted images in Fig. 3, there is no visual information observed in the encrypted image.

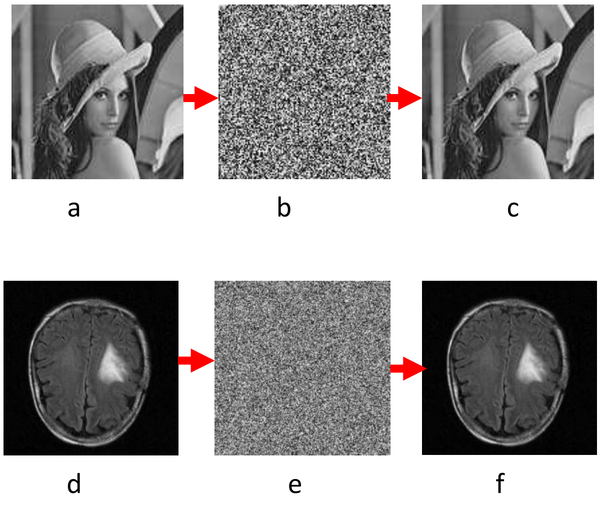

Fig. 3. Frame (a) and (d) Gray image show the original image of Lena and IRM, frame (b) and (e) respectively show the encrypted image, frame (c) and (f) respectively show the decrypted image.

a b

Fig. 4, Frame (a) show the difference between original image figure 3(a) and decrypted image figure 3(c). Frame (b) show the difference between original image figure 3(d) and decrypted image figure 3(f).

Difference between original images and their decrypted images shown in figure 3, are illustrated in figure 4(a), 4(b), are prove that, there is no loss of information, the difference is always 0.

5. Security Analysis

5.1. Key Space Analysis

The key space should be large enough to make the exhaustive search attack infeasible. Since the algorithm has a chain 1024bits is the initialization of 13 registers, the intruder needs 2^{1024} tests by exhaustive search. An image cipher with such as a long key space is sufficient for reliable practical use.

5.2. Histogram Analysis

In the experiments, the original images and its corresponding encrypted images are shown in figure 3, and their histograms are shown in figure 5. It is clear that the histogram of the encrypted image is nearly uniformly distributed, and significantly different from the histogram of the original image. So, the encrypted image does not provide any clue to employ any statistical attack on the proposed encryption of an image procedure, which makes statistical attacks difficult.

These properties tell that the proposed image encryption has high security against statistical attacks. In the original image (i.e. plain image), some gray-scale values in the range [0, 255] are still not existed, but every gray-scale values in the range [0, 255] are existed and uniformly distributed in the encrypted image. Some gray-scale values are still not existed in the encrypted image although the existed gray-scale values are uniformly distributed. Different images have been tested by the proposed image encryption procedure.

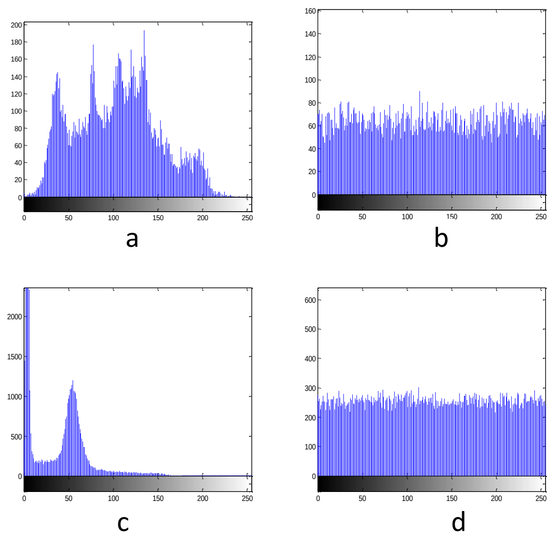

Fig. 5, Histogram analysis: Frame (a) and (c) respectively, show the histogram of the plain images shown in figure 3(a) and 3(d). Frame (b) and (d) respectively; show the histogram of the encrypted images shown in figure 3(b) and 3(f).

5.3. Correlation Coefficient Analysis

Correlation is a measure of the relationship between two variables if the two variables are the original image and their encryptions then they are in prefect in correlation. In this case the encrypted image is the same as the original image and the encryption process failed in hiding the details of the original image. If the correlation coefficient equals zero, then the original image and its encryption are totally different i.e. the encryption image has no features and highly independent on the original image. If the correlation coefficient equal -1, this means encrypted image is a negative of the original image.

Table 1 gives the corresponding correlation coefficient between plain-images (i.e. original image) shown in figure 3(a), 3(d) and 6(a) and their encrypted images. It is observed that the correlation coefficient is a small correlation between plain-images and encrypted image.

Table 1. Correlation Coefficients analysis

Cases	Correlation coefficient
Image 3.a	-0.0086
Image 3.d	-0.0068
Image 6.a	-0,0055

5.4. Image Entropy

A secure cryptosystem should fulfill a condition on the information entropy that is the ciphered image should not provide any information about the plain image. It is well known that the entropy $E(m)$ of a message source m can be calculated as:

$$E(m) = \sum_{i=0}^{G-1} p(m_i) \log_2 \frac{1}{p(m_i)} \tag{4}$$

where G Gray value of an input image (0-255), $p(m_i)$ represents the probability of symbol m_i and the entropy is expressed in bits. Let us suppose that the source emits 2^8 symbols with equal probability, i.e., $m = \{m_1, m_2, ..., m_{2^8}\}$ Truly random source entropy is equal to 8. Actually, given that a practical information source seldom generates random messages, in general its entropy value is smaller than the ideal one. However, when the messages are encrypted, their entropy should ideally be 8. If the output of such a cipher emits symbols with entropy less than 8, there exists certain degree of predictability, which threatens its security.

Table 2 gives the entropy values of plain images and of their encryptions images shown in figure 3 and 6. The values obtained are very close to the theoretical value of 8. This means that information leakage in the encryption process is negligible and the encryption system is secure upon the entropy attack.

Table 2. Image Entropy

Plain-Image	Entropy	Encrypted Image	Entropy
Image 3.a	7.4697	Image 3.b	7.9870
Image 3.d	5.4753	Image 3.e	7.9971
Image 6.a	2.8284	Image 6.c	7.9889

5.5. Key sensitivity analysis

An ideal image encryption procedure should be sensitive with respect to secret key. The change of a single bit in the secret key should produce a completely different encrypted image. To prove the robustness of the proposed scheme, sensitivity analysis with respect to key is performed. High key sensitivity is required by secure image cryptosystems, which means the cipher image cannot be decrypted correctly even if there is only a small difference between the encryption and decryption keys. In the key sensitivity tests, we change one bit of the key. Figure 6 show key sensitivity test result. It can be observed that the decryption with a slightly different key (different secret key or initial values) fails completely. Therefore, the proposed image encryption scheme is highly key sensitive.

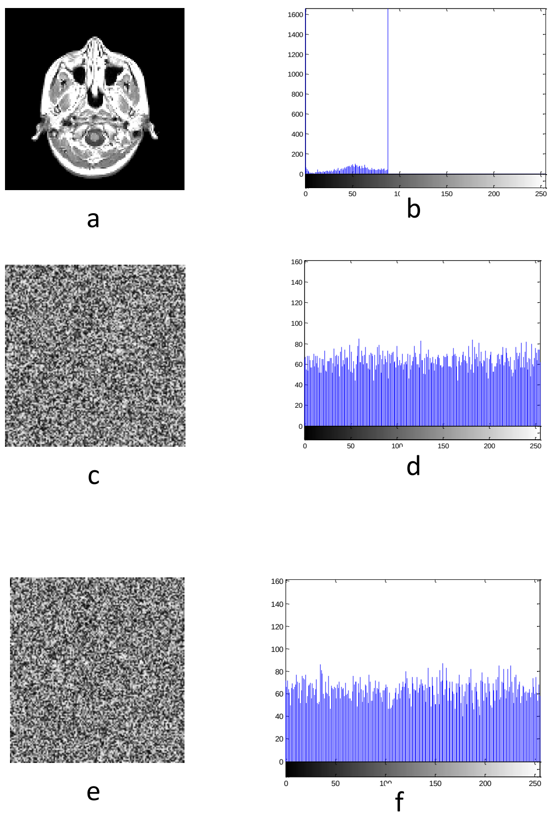

Fig. 6, Sensitivity analysis: (a) original mage of mri, (b) histogram of mri (c) encrypted image by a 1024 bits key, (d) histogram of encrypted image by a 1024 bits key, (e) decrypted image by key in (b) with a bit changed,(f) histogram of decrypted image by key in (b) with a bit changed.

5.6. Noise analysis

We also tested the resistance our cryptosystem to the noise by adding to the cipher-images a noise. From the cipher-images illustrated in the figures 3.b, and 3.e we added a noise of the same size of plain-images. The results are given in the figure 7a and 7.c. From the images 7a and 7.c, we apply the decryption algorithm presented in section 3.4; we have the results illustrated in figure 7b and 7.d. The noise added to ciphers-images 3.b, and 3.e is a matrix containing pseudo-random values drawn from a normal distribution with mean zero and standard deviation one, generates with function “randn”. In this case examined, we can note that the decrypted images presented in figures 7b and 7.d are identical to the original images (see 3.a, 3.d), there is no difference pixel with pixel has indeed between the decrypted images and plain-images because of reversibility of our technique of encryption.

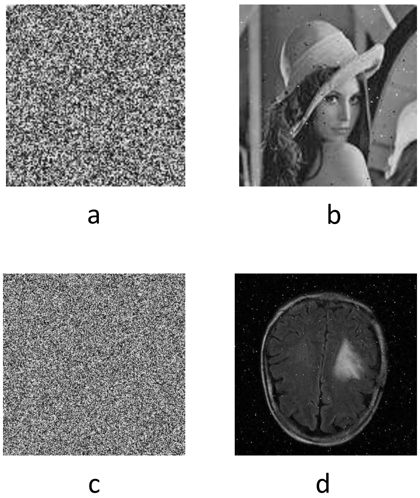

Fig. 7, Noise Analysis: Frame (a) show cipher image shown in figure 3 (b) with noise added , Frame (b) show deciphered image, Frame (c) show cipher image shown in figure 3 (e) with noise added , Frame (d) show deciphered image.

5.7. Berlekamps-Massey

The Berlekamps-Massey attack [17] requires $2\lambda(s)$ data successive. In order to mount a Berlekamp-Massey attack, the key stream generator must produces a key stream with linear complexity highest possible. This linear complexity depends entirely on the combining function. Linear complexity $\lambda(s) = f(53, 59, ..., 101, 102)$ used in our cryptosystem is between 2^{47} and 2^{48} , it is sufficiently large. This complexity completely excludes to use the Berlekamp-Massey attack.

5.8. Correlation Attack

The combining function f used in our system is correlation immune of order 5. In order to mount a correlation attack of Sigenthaler [18], the attacker must consider at least six shift registers simultaneously. The sum of the lengths of the shortest six LFSRs of the keystream generator is 53+59+61+67+71+73=384. Therefore, the complexity of Siegenthaler's correlation attack against our approach is at least $\mathrm{O}\left(2^{384}\right)$. This is out of reach this type of attack.

5.9. Algebraic Attack

In the algebraic attacks, the system is rewritten in the form of a nonlinear system of equations between the output of the filtering function f and its inputs in the following way:

$z_0 = f(K)$;

$z_1 = f\left(h(K)\right)$;

… ;

$z_i = f\left(h^i(K)\right)$,

Here h denotes the linear update function to the next state of the LFSR's involved, K the total key of the system. Complexity to solve this system of equations strongly depends on the degree of these equations. The complexity $C\left(L,d\right)$ of the algebraic attack on the stream cipher system with a key of size L bits and equations of d degree is given by $C\left(L,d\right) = \left(\sum_{i=0}^{d} \binom{L}{i}^{w}\right) = L^{w.d}$, where w corresponds to the coefficient of the method of the solution most effective by the linear system and d is equal to algebraic immunity of the function of combination. We employ here the expression of Strassen [19] which is $w = \log_2(7) \approx 2.807$. In our cryptosystem the secret key is 1024 bits and the algebraic immunity of the nonlinear filter function is equal to 6. This leads to an algebraic attack with a complexity which is between 2^{168} and 2^{169}, which is sufficiently large. It is not easy to make a linear approximation of the nonlinear filter function within the framework of an algebraic attack.

6. Conclusion

In this paper, an encryption scheme using stream cipher based on nonlinear combination generator presented. The proposed encryption system included two major parts, 13 LFSRS with maximum period whose length are all pairwise comprimes, the goal of this engine is to provide one or more continuations, having good statistical properties already and nonlinear Boolean function satisfying all the cryptographic criteria necessary carrying out the best possible compromises. Simulations were carried out different images. The encrypted images obtained for these input images and the corresponding histograms are discussed. It is seen that encrypted images does not have residuals information and the corresponding histograms are almost flat offering good security for images. The proposed schemes key space is large enough to resist all kinds of brute-force attack. In addition, this method is very simple to implement, the encryption and decryption of image. Here the security aspects like key space, statistical and sensitivity with respect to key are discussed with examples. It is seen that the present cryptosystem is secure against the statistical, brute force and cryptanalytic attacks.

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NEW TRENDS IN MATHEMATICAL SCIENCES
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The Relationship Between Some Kinds of Ideal in The Order

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Abstract: This work will discuss one of the structures in Mathematics Algebra, namely Order. Simply put, order is a ring that certain criteria. For R is a ring which is of order, defining the R-ideal is difference with defining ideal (regular) in R as it is known in general. An R-ideal in R is certainly an ideal (regular) in R. However, in general, an ideal (regular) in R is not an R-ideal in R. However, in certain circumstances, the ideal (regular) in R is also an R-ideal. In addition to R-ideal, in order also known notion some other ideal. In this paper will be discussed the relationship between several types of ideal in the order.

Keywords: ideal, invertible, order, reflexive, quotient ring.

1. Introduction

This paper will discuss one of the structures in Mathematics Algebra, that is Order. Further, some kind of ideal that is closely associated with the order and types of linkages between these ideals will be the focus of study. Simply put, the order is a ring that meets certain criteria. For defining the order necessary for the understanding of the quotient ring and some other sense. Moreover, the definition of the quotient ring requires understanding of regular elements. Therefore, the order begins with understanding the definition of a regular element in a ring.

In the ring R which is an order known some ideal sense, such as R-ideal, fractional R-ideal, reflexive ideal, invertible ideal, and v-ideal. Furthermore, for R is a ring which is an order, the definition of R-ideal in R different from the defining ideal (regular) in R as it is known in general. An R-ideal in R is a ideal (regular) in R. However, in general, an ideal (regular) in R is not a R-ideal in R.

This paper will describe the notion of ideal types referred to in paragraph above. Apart from presenting the ideal type, is presented as well as some theory that links between the order and these ideals.

2. Definition, symbol, and Basic Theory

This study is a literature review of studies that use methods of adaptation and exploitation. Therefore, in this section are presented some sense, the basic theories, and the results of studies of several researchers who will adapted and exploited.

Definition. 2.1 [Zariski dan Samuel, 1958]

Let R be a ring. An element $0 \neq x \in R$ is called right regular if $xr = 0$ implies $r = 0$. While the left regular element is defined similar. If $x \in R$ is a right and left regular element, then $x \in R$ is called reguler.

The set of all regular elements in a ring form a set which is closed under multiplication and this set contains the identity element of R. The set is called multiplicative set. In general, a subset of a ring which is closed under multiplication, contains the identity element, and does not contain zero element is called multiplicative set.

Reguler elements in a ring does not necessarily have an inverse in the ring. This encourages the undefined quotient ring, which ring contains elements that revert all regular elements with specific propries.

Let Q be a ring that contains the ring R and the inverse of all regular elements in R. The ring Q is called the right quotient ring of R, if every $q \in Q$ can be written $q = rs^{-1}$ for an $r \in R$ and s is a regular element in R. The right quotient ring of R is defined similar. A ring Q arena called the quotient ring of R if Q is a right and left quotient ring of R.

Furthermore, the ring which is the quotient ring of the ring itself is called the quotient ring. Thus, it can be concluded that a ring Q is called the quotient ring, if every regular element is a unit element.

Observing the process of defining the quotient ring of a ring, it appears that not every ring has a quotient ring. Associated with the existence of the quotient ring, there are necessary and sufficient condition of a ring which has a quotient ring. Terms are granted by understanding the conditions Ore.

Let S be a subset of the ring R which is closed under multiplication. The set S is said to satisfy the right Ore condition if, for each $r \in R$ and $s \in S$ there exist $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$. Left Ore condition is defined similar. Furthermore, the ring R which satisfy the right (left) Ore condition for $S = R$ is called right (left) Ore ring.

Using the Ore condition above, the following necessary and sufficient conditions are presented ring that has a quotient ring.

Lemma 2.1 [McConnell and Robson, 1987]

1. A ring with identity element which does not contain divisor of zero element has a right quotient ring if and only if it is a right Ore domain.
2. A right Noetherian ring with identity element which does not contain divisor of zero element is a right Ore domain.

Using Lemma 2.1 we can conclude that the right Noetherian ring with identity element which is not contain divisor of zero elements has a quotient ring.

Furthermore, relooking at the quotient ring, it was found that two different ring may have the same quotient ring. For example, the ring $k[x]$ and $k[x, x^{-1}]$. This phenomenon inspired the definition of order.

Let Q be the quotient ring. Subring $R \subseteq Q$ is called the right order in Q if every $q \epsilon Q$ in the form $q = rs^{-1}$ for some $r, s \in R$. So also for the order left, subring $R \subseteq Q$ is called the left order in Q if every $q \epsilon Q$ in the form $q = s^{-1}r$ for some $r, s \in R$. If R is a right order once the left order, then R is called an order.

In the quotient ring, order is not unique. This encourages defines the maximum order.

Definition 2.2 [McConnell dan Robson, 1987]

Let Q be a quotient ring and $R_1, R_2 \subseteq Q$ are right orders in Q. Relation $\sim$ is defined with $R_1 \sim R_2$ if there exist $a_1, a_2, b_1, b_2 \in Q$ unit in Q such that $a_1 R_1 b_1 \subseteq R_2$ and $a_2 R_2 b_2 \subseteq R_1$.

It is clear that the relation ~ in Definition 2.2 is an equivalence relation. These relationships will form the equivalent classes. Order right order R is called right-maximal if R maximum in the equivalent class. Similar maximal left order defined. While R is called maximal order if R is a maximal order right and left.

Several types of order are defined in the order or closely related to the order presented in this section. Ideal types of order in question, among others, fractional ideal, invertible ideal, and v-ideal. Apart from presenting the ideal type, is presented also some theories that found links between the order and these ideals.

Definition 2.3

Let R be an order in the quotient ring Q. Right submodule I of Q_R that meet $aI \subseteq R$ and $bR \subseteq I$ for some unit $a, b \subseteq Q$ is called fractional right R-ideal. Fractional left R-ideal is defined similar. If I is a left and right R-ideal fractional, then I called a fractional R-ideal. Furthermore, if I is an R-fractional ideal right and $aI \subseteq R$, then I called right R-ideal. The same is true for left R-ideal. R-ideal that once left and right R-ideal is called R-ideal.

Using Definition of R-ideal above is not the same as the defining ideal (regular) in R as it is known in general. An ideal (regular) I in R is not necessarily a R-ideal in R, because the unit element $b \in Q$ that satisfy $bR \subseteq I$ do not necessarily exist. However, in certain circumstances, the ideal (regular) in R is also an R-ideal.

Here, some definitions and notations used in the theory of order. Suppose that R is order in the ring Q. For the sets of X and Y of Q, is defined (Marubayashi, Miyamoto, and Ueda, 1997),

$$(X, Y)_r = \{q \in Q \mid Yq \subseteq X\}$$

$$(X, Y)_l = \{q \in Q \mid qY \subseteq X\}$$

$$X^{-1} = \{q \in Q \mid XqX \subseteq X\}.$$

For right fractional R-ideal I of Q, denoted

$$O_r(I) = (I:I)_r = \{q \in Q \mid Iq \subseteq I\}.$$

For left fractional R-ideal I of Q, denoted

$$O_l(I) = (I:I)_l = \{q \in Q \mid qI \subseteq I\}.$$

They are called right order and left order of I respectively.

Using the above definitions and notation, the relationship between the maximum order, fractional ideal, and R-ideal is given in the following theorem.

Theorem 2.2 [McConnell dan Robson (1987)]

If $R$is a right order in Q then the following conditions are equvqlent:

a. R is a maximal right order
b. $O_r(I) = O_l(I) = R$ for all fractional R-ideal I.
c. $O_r(I) = O_l(I) = R$ for all R-ideal I.

Fractional ideal, as defined in Definition 2.3, was further developed into an invertible ideal and v-ideal.

Definition 2.4 [Marubayashi, Miyamoto, dan Ueda, 1997]

A fractional R-ideal I is called right v-ideal if $I_v = I$ where $I_v = (R:(R:I)_r)_l$. Similarly, fractional R-ideal J is called left v-ideal if ${}_vJ = J$ where ${}_vJ = (R:(R:I)_l)_r$. A fractional R-ideal I is called v-ideal, if $I_v = I = {}_vI$. Meanwhile, a fractional R-ideal I is called invertible if $(R:I)_l I = R = I(R:I)_r$.

Apart from the invertible ideal and v-ideal, fractional ideal can also be developed into a reflexive ideal. To define the following notation is required reflexive ideal. Suppose R is a right order in the quotient ring Q and I is a fractional right R-ideal, denoted

$$I^* = (R:I)_l = \{q \in Q \mid qI \subseteq R\}.$$

Apart from the notations, the following theorem is needed to clarify the definition of reflexive ideal.

Theorem 2.3 [McConnell and Robson, 1987]

If R and R' are maximal orders in quotient ring Q and I is a fractional R-ideal, then $(R: I)_l = (R': I)_r$.

Using Theorem 2.3 and the notation I^*, the reflexive ideal is expressed as follows.

Definition 2.4 [McConnell and Robson, 1987]

Let R be an order in the quotient ring Q and I be a fractional R-ideal. If $I = I^{**}$, then I is said reflexive.

Observing the sense of reflexive ideal and *v*-ideal, the relationship between them is obtained as shown the following lemma.

Lemma 2.4

R be an order in the quotient ring Q and I be a fractional R-ideal. Then I reflexive if and only if I is a v -ideal.

Proof:

Using Theorem 2.2, we obtain

$$(R: (R: I)_l)_l = I = (R: (R: I)_r)_r.$$

On the other hand, Theorem 2.3 stats that $(R: I)_l = (R: I)_r$. Therefore we obtain the following:

$(R: (R: I)_l)_l = I$ if and only if $(R: (R: I)_l)_r = I = (R: (R: I)_r)_l$.

This completes the proof. ■

Lemma 2.4 has presented the link between *v*-ideal with a reflexive ideal. In addition to the reflexive ideal, it turns out, *v*-ideal is also associated with invertible ideal. To prove the links between them, the following lemma is required.

Lemma 2.5 [Marubayashi, Miyamoto, dan Ueda, 1997]

If I is an invertible ideal, then $(R: I)_l = I^{-1} = (R: I)_r$.

Furthermore, the linkage between the invertible ideal with *v*-ideal is given in the following lemma and it can be proved using Lemma 2.5.

Lemma 2.6

Let R be a ring with identity element and I is an ideal in R. If I is an invertible ideal, then I is a *v*-ideal.

Proof:

Let I be an invertible ideal, then $(R: I)_l = I^{-1} = (R: I)_r$. For $q \in I$, $(R: I)_l q = I^{-1} q \subseteq R$. This means $q \in I_v = (R: R: I)_l)_r$. So $I \subseteq I_v$. Conversely, let $q \in I_v$, then $q \in I$. So $I_v \subseteq I$. Therefore we get $I_v = I$. with similar way, we can show that ${}_vI = I$. This implies ${}_vI = I = I_v$ or I is a v-ideal. ■

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Numerical solving for nonlinear using higher order homotopy Taylor-perturbation

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Abstract: Rootfinding is a classical problem that still remains an interest to many researchers. A series of hybrid methods called Higher Order Homotopy Taylor-perturbation method via start-system functions (HTTPss) are implemented to give approximate solutions for nonlinear equations, $f(x) = 0$. The techniques serve as alternative methods for obtaining approximate solutions for different types of nonlinear equations. Thus, this paper presents an analysis on numerical comparison between the classical Newton Raphson (CNR), Homotopy Perturbation method (HTPss) and Higher Order Homotopy Taylor-perturbation via start-system (HHTPss). A computational system Maple14 is used for this paper. Numerical and Illustrative results reveal that HHTPss methods are acceptably accurate and applicable.

Keywords: HHTP, NONLINEAR, START SYSTEM

1. Introduction

Numerical method provides as an alternative way to solve nonlinear problems in many areas of science and technology. Among the numerical methods known for its efficiency and effectiveness is the Newton-Raphson method which is proven for its second order convergence. However there are problems in this method because its effectiveness lies in the accuracy and closeness of the initial value picked at the beginning of the iteration process.

There are also modified Newton methods for problem such as multiroots where a multiplier ‘m’ representing the highest power of higher order polynomial functions, is added into the original Newton-Raphson iterative method (S.G. Li et.al, 2010; Rafiq & Awais, 2008). However, it is effective only for simple nonlinear multiroots problems, but ineffective for more complex equations and higher order homotopy Taylor-perturbation. Besides the easiness of Newton-homotopy, it does not guarantee to converge. Other new and surprising methods also offer solutions to problems such as the perturbation technique that is based on an assumption that a small parameter must exist in the equation, the homotopy method (He, 1999, 2009), the hybrids such as homotopy perturbation (hpm), higher order hpm and hpm and with startsystem, as well as the robust approaches of these methods (Saeed et.al., 2011; Nor Hanim et.al, 2011a-d; Palancz, 2010; Saeed & Khthr, 2010; Pakdemirli & Boyaci, 2007). The applications of the start-system functions helps to accelerate the rate of convergence of the functions, because of the closeness of the start-system values suggested (Nor Hanim et.al, 2011 1-c). Their approaches are iterative, and in many cases appears to be considerably more computational oriented. The ideas of these new approaches are somehow simple and proven effective

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This paper presents hybrid methods of higher order Taylor's series, perturbation techniques, homotopy continuation method and the start-system concepts in order to generate faster and more effective ways to solve multiroots of nonlinear problems for $f(x) = 0$.

2. Methodology

Higher-order homotopy Taylor-perturbation method involves the substitution of the perturbation techniques into the Taylors series up to the value of 'n' required, and the conversion of the original functions into homotopy functions (Nor Hanim et.al, 2011a-d, 2010). Furthermore, the convex homotopy for the function is defined as $H(x,\lambda): \Re \times [0,1] \rightarrow \Re$ as, $H(x,\lambda) = (1-\lambda)p(x) + \lambda q(x) = 0$; where, λ is an embedded parameter, $\lambda \in [0,1]$; $p(x)$ as the start system function where $p(x) = x^n - C$, and n is preferably the highest power of x of a nonlinear function $f(x)$; $q(x)$ as the target system function; C is any real number in $f(x)$; and $H(x,0) = p(x)$ & $H(x,1) = q(x) = f(x)$. Here, the step size is set to 0.2 and the stop-criteria are set to $|x_{i+1} - x_i| < 10^{-5}$.

To determine the initial value x_0, only equate $p(x)$ to zero. The selection of $p(x)$ only requires a part of the original equation $f(x)$, which is known to have at least one trivial solution. There are also several other ways to identify a start-system of a linear homotopy as mentioned by Nor Hanim et.al. (2011 a-d) and Palancz et.al. (2010).

2.1. First Order Taylor-perturbation

The formulation of the first order Taylor-perturbation is illustrated. For a nonlinear equation, let $f(x) = 0$. Assume a perturbation expansion with only one correction term $x = x_0 + \varepsilon x_1$. By using algebraic manipulation $x - x_0 = \varepsilon x_1$, insert into Taylor Series expansion of order 1,

$$P_1(x) = f(x - x_0) \cong f(x_0) + f'(x_0)(x - x_0). \tag{1}$$

Substitute $\varepsilon x_1 = x - x_0$ into (1)

$$P_1(x) = f(\varepsilon x_1) \cong f(x_0) + f'(x_0)(\varepsilon x_1). \tag{2}$$

Assuming the RHS equals to 0 then becomes, and solve for εx_1, and we get

$$\begin{array}{c} f(x_0) + f'(x_0)(\varepsilon x_1) \cong 0 \\ \varepsilon x_1 = -f(x_0)/f'(x_0) \end{array} \rightarrow x = x_0 - f(x_0)/f'(x_0). \tag{3}$$

Add an iteration feature such as follow (equivalent to the classical Newton-Raphson),

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \quad n = 0,1,2,3 \ldots \tag{4}$$

2.2. Second Order Taylor-perturbation (HTP)

The formulation second order Taylor-perturbation is illustrated. For a nonlinear equation, let $f(x) = 0$. Assume a perturbation expansion with two correction term $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2$, thus using algebraic manipulation we get,

$x - x_0 = \varepsilon x_1 + \varepsilon^2 x_2$. (5)

Inserting (5) into Taylor Series expansion of order 2,

$$P_2(x) = f(x - x_0) \cong f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2. \tag{6}$$

Substitute $\varepsilon x_1 + \varepsilon^2 x_2 = x - x_0$ into (6),

Similarly, expand and ignore the last 2 terms and factorize, we get

$$f(x_0) + f'(x_0)\varepsilon x_1 + f'(x_0)\varepsilon^2 x_2 + f''(x_0)\varepsilon^2 x_1^2/2 \cong 0, \tag{7}$$

$$f(x_0) + \varepsilon x_1 f'(x_0) + \varepsilon^2(x_2 f'(x_0) + x_1^2 f''(x_0)/2) \cong 0. \tag{8}$$

The sum of first 2 terms and the sum of last 2 terms of equation (8) be equal to 0,

$$f(x_0) + \varepsilon x_1 f'(x_0) \cong 0, \\ \varepsilon^2 (x_2 f'(x_0) + x_1^2 f''(x_0)/2) \cong 0. \tag{9}$$

From equation 9, solve εx_1 and $\varepsilon^2 x_2$. Hence we get,

$$\varepsilon x_1 f'(x_0) = -f(x_0) \\ \therefore \varepsilon x_1 = -f(x_0)/f'(x_0); \tag{10}$$

$$\varepsilon^2 x_2 f'(x_0) + \varepsilon^2 x_1^2 f''(x_0)/2 \cong 0 \\ \varepsilon^2 x_2 f'(x_0) = -\big(-f(x_0)/f'(x_0)\big)^2 f''(x_0)/2 \\ \therefore \varepsilon^2 x_2 = -\big(f(x_0)\big)^2 f''(x_0)/2\big(f'(x_0)\big)^3; \tag{11}$$

Then, substitute equation (10) and (11) into equation (8),

$$x = x_0 - f(x_0)/f'(x_0) - f''(x_0)\big(f(x_0)\big)^2/2\big(f'(x_0)\big)^3. \tag{12}$$

Add an iteration feature such as follow, (The iteration scheme of HTP)

$$x_{n+1} = x_n - f(x_n)/f'(x_n) - f''(x_n)\big(f(x_n)\big)^2/2\big(f'(x_n)\big)^3;\ n = 0,1,2,3\ \ldots \tag{13}$$

Finally, convert equation (13) to homotopy equation,

$$x_{n+1} = x_n - H(x_n)/H'(x_n) - H''(x_n)\big(H(x_n)\big)^2/2\big(H'(x_n)\big)^3;\ n = 0,1,2,3\ \ldots \tag{14}$$

The same steps of calculations were done for order-3, order-4, order-5 and order-6. The equations were summarized as in Table 1. Hereafter, our discussion will only proceed with the above schemes. The derivations were first done manually. Then, countercheck using the mathematical software, *Maple14*.

Table1. The iteration scheme of the higher order correctional terms of homotopy Taylor-perturbation (HHTP) methods with start-system (ss).

Correctional Terms	**Higher Order Homotopy Taylor-Perturbation Method (HHTP-iterative form, $\boldsymbol{x = x_{(i+1)}}$)**
1st order	$x_i - H(x_i,\lambda)/H'(x_i,\lambda)$
2nd order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$
3rd order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$ $+ H^3(x_i,\lambda).\big[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\big]$
4st order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$ $+ H^3(x_i,\lambda).\big[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\big]$ $+ \big[H''^2(x_i,\lambda).H^4(x_i,\lambda)/4.H'^6(x_i,\lambda)\big]$
5st order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - \big[H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)\big]$ $+ H^3(x_i,\lambda).\big[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)\big]$ $+ \big[H''^2(x_i,\lambda).H^4(x_i,\lambda)/4.H'^6(x_i,\lambda)\big]$ $- \big[H''^2(x_i,\lambda).H^5(x_i,\lambda).[H'''(x_i,\lambda) - 3.H'(x_i,\lambda)]/2.H'^8(x_i,\lambda)\big]$

6st order	$x_i - [H(x_i,\lambda)/H'(x_i,\lambda)] - [H''(x_i,\lambda).H^2(x_i,\lambda)/2H'^3(x_i,\lambda)]$ $+ H^3(x_i,\lambda).[H'(x_i,\lambda)H'''(x_i,\lambda) - 3.H''^2(x_i,\lambda)/6H'^5(x_i,\lambda)]$ $+ [H''^2(x_i,\lambda).H^4(x_i,\lambda)/4.H'^6(x_i,\lambda)]$ $- [H''^2(x_i,\lambda).H^5(x_i,\lambda).[H'''(x_i,\lambda) - 3.H'(x_i,\lambda)]/2.H'^8(x_i,\lambda)]$ $+ H^2(x_i).H''(x_i)$ $/6.H'(x_i)^{10}.\{[(H'''(x_i).H''(x_i)^2H(x_i)^4/2)$ $+ ((H(x_i)/2H^2(x_i)^2) \times (H'(x_i).H'''(x_i) - 3H'(x_i)^2))^2]$ $+ [((3/4)f''(x_i)^3.f(x_i)^4)]\}$

Sources:(Nor Hanim et.al, 2011a-d)

3. Numerical Analysis

The list of the higher order iterative schemes of Homotopy Taylor-perturbation by using start-system (HHTPss) can be referred at Table 1. While, Table 2 shows the selected nonlinear equations, start-system functions, $p(x)$, and the selected initial value, x_0. The efficiency of the iterative hybrid methods from 1st order to the 5th order homotopy Taylor-perturbation method using start-system (HHTPss) is also given in Table 2, which gives equal or better results in terms of convergence rate as compared to the classical Newton-Raphson. The given test function (i)-(xii) are used and the results of the approximated zeros is given in 10^{-5} error accuracy. Furthermore, the choice of a suitable $p(x)$ is not unique and different choices of $p(x)$ work better for different types of equations.

Table 2: The approximated zeros using Classical Newton-Raphson (CNR) and higher order Homotopy Taylor-Perturbation (HHTP) via start-system: startsystem function, initial value and number of iterations needed to converge.

	NL Functions $q(x) = f(x)$; $p(x)$; $p(x) = 0 = x_0$	**CNR**	**1st**	**2nd**	**3rd**	**4th**	**5th**
i	$\sin(3x) + 4(e^{2x}) - e^{(-x)} + \ln(2x) - 3$ $(e^{(x)} + -3)$ 1098612289	5	2	2	1	4	5
ii	$\frac{1}{2} + \frac{1}{4}x^2 - \sin x - \cos(2x)$ (x^2) 0.5	3	1	2	1	1	1
iii	$x^2 - 4\tan(2x) - \tanh(x)$ $(\tanh(x))$; 0.0	4	1	1	1	1	1
iv	$(e^x + x^2 - 2)^4$ $(x^{16} - 16)$ -1.189207115,1.189207115	37	34	24	20	20	20
v	$(2x^2 - 9)^3.(\ln(x))$ $(x^2 - 9)$ -3.0,3.0	27	21	15	13	11	11
vi	$(\tan(x) - \tanh(x))^5$ $\tan(x)$ 0.0,-2.514866859, 2.514866859*	118	39	28	24	24	23
vii	$(x^5 - x^3 - 2)^3$ $(x^5 - 2)$ 1.148698355	21	19	14	12	12	12
viii	$(e^x - \sin(x))^2$ $(\sin(x))$; 0.0,-3.0*	14	12	9	8	8	8
ix	$(xe^x + 2^{-x} + 2\cos(x) - 6)^2$	3	1	1	1	1	1

	$(\cos(x))$ 1.570796327						
x	$(x-2)^2.(x^4+6x-40)$ (x^4-40); 2.514866859,2.514866859	6	5	3	3	3	3
xi	$(x-2)^2(\sin(x))$ (x^3-2); 1.259921050	4	2	2	2	2	2
xii	$(x^2-e^x-3x+2).(\cos(x))$ (x^2-3x);0.0, 3.0	3	2	1	1	1	1

4. Conclusions

It is concluded that the method of higher-order homotopy Taylor-perturbation is an effective alternative method in accelerating the converge rate in solving nonlinear functions. Combined with the method of start-system, it facilitates the way to determine the initial value for the iteration. Thus, the computing time is reduced.

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New Similarity Measure for Shape Recognition

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Abstract: Blob shape recognition is used in various application fields such as industrial vision systems, parts recognition, positioning, inspection, etc. This is based on numerical signature which is an effective image processing technique for shape recognition.

This paper describe a new fast algorithm for pattern recognition, shapes are being coded by using two vectors, the first one is contour distances and the second one is of the surface histogram, these are working in relative manner. The principle coding is reduced to a product between the mask and the binary image of the shape.

The first merit of this algorithm is the high-speed image processing, instead of using images it does operate on vectors. The second merit is the precise recognition of known geometries shapes even for arbitrary or complexes ones.

Keywords: Distance mask, Histogram equality, Numerical signature, linear approximation, Stationary series, Pattern recognition

1. Introduction

The pattern recognition technique is a very important task and very required in various industrials systems and in vision systems; such as in positioning, automated visual inspection and many other applications.

In these applications the processing time is very important and high speeds are much sought for the industrial requirements. But the image has a large numerical data which means a conventional basic processing is too slow. In order to speed up the processing there are two types of solutions. The first type consists of subdividing the processing into tasks which will be given to parallel processors [1]. Other solution consist of reducing the quantity of information and instead of working on the whole data of the image; we use a compressed or reduced quantity [2].

Our work is based on the second type of solution, because the image immediately coded in two vectors, the first vector having the dimension of the contour pixels number of the object in question, the second vector having the dimension of the circle radius generating this object, then all the operations are deduced from the manipulations of the two vectors. In addition, the operation of information compression is very fast as we are going to present, this has been leading to inexpensive method in time and very reliable. Thus, this will find implementation in visual recognition real time process.

In this paper, we will present first the theoretical tools used. Then, we will use these tools to present the information compression principal and the image coding. Later this presentation will be used in detailed manner to expand our steps of recognition. Finally we will present the obtained experimental results over several samples of images.

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2. Theoretical Tools

From the viewpoint of shape recognition, the shape plane, this can be described by the couple of information; its surface S and its contour C. This information can be given by the Cartesian coordinate as follows:

1. A set of Cartesian coordinate point (x, y) such that $(x, y) \in S$.
2. An analytical function f describing C such that $y = f(x)$.

with $x \in D$

D: The definition domain of the shape,

x: The departure set of points of D,

y: The arrival set of points.

The original image data is two dimensional matrix therefore this shape will be presented by a set of finite couple. If the contour C is constituted of n pixels the a couple vector of (x_i, y_i) with $i = 1..n$ of dimension n sufficient to represent it $C = \{(x_1, y_1), (x_2, y_2), \dots, (x_i, y_i), \dots, (x_n, y_n)\}$.

Or in the coordinate system (d, θ), $C = \{(d_1, \theta_1), (d_2, \theta_2), \dots, (d_i, \theta_i), \dots, (d_n, \theta_n)\}$. Thus, C is represented by two sets:

1. The set of distances

$$V = \{d_1, d_2, \dots d_i, \dots d_n\} \tag{1}$$

2. The set of angles $\{\theta_1, \theta_2, \dots \theta_i, \dots \theta_n\}$ which will be replaced by a set pixels forming C.

This representation of shapes offers two possibilities of exploitation. Handling these distances to sort an analytical expression $y = f(x)$ modeling the contour C, which is valid for known geometries shapes. But for arbitrary or complexes shapes the modeling becomes difficult, thus we were interested to working directly from this numerical representation of shapes without going through the analytical expression.

A planer shape can undergo translations, rotations and homothety *zoom*. If we denote by S_{before}, C_{before} the surface and the contour before transformation and S_{after}, C_{after} after then,

$$S_{after} = k^2 . S_{before} \cdots and \cdots C_{after} = k . C_{before} \tag{2}$$

with $k = 1$ for translations and rotations and $\neq 1$ for homothety.

We start by coding the contour by the distances vector (1) C_{before} by V and C_{after} by W, therefore for the translations and rotations we have

$$V = W \Longrightarrow \|V\| \simeq \|W\| \tag{3}$$

Whereas for homothety of k ratio:

$$\|V\| \simeq k * \|W\| \tag{4}$$

For $k < 1$, V and W are of the type:

$$V = \{d_1, d_2, \dots d_i, \dots d_n\} \dots W = \{d_{11}, \dots d_{1k}, d_{21}, \dots d_{2k}, \dots \dots \dots d_{n1}, \dots d_{nk}\} \tag{5}$$

Such as,

$$d_1 = k * d_{11}, d_2 = k * d_{21}, \dots \dots d_n = k * d_{n1} \tag{6}$$

Inversely if $k > 1$.

At this stage, we must point out the following problems:

1. The ratio k is not always an integer which gives a number of pixels $d_{i1}d_{i2} \dots d_{ik}$ not necessarily an integer and which should be an integer due to the digital mesh?

2. The pixels $d_{i1}d_{i2} \dots d_{ik}$ of W have only one image which is d_i in V that we have associated d_{i1}, then how about the other points $d_{i2} \dots d_{ik}$?

For the first problem when we are on the element d_i of rank i in V, we take the rounded product $j = round(k * i)$ to determine its counterpart for d_j in W. This allow us some stationary, hence a frequency of repetition more or less regularly. Concerning the second problem, this consists of giving estimation of missing distances $d_{i2} \dots d_{ik}$ that have not images in V, we have opted for a linear approximation of first order, for example having;

$$d_1 = k * d_{11} \wedge d_2 = k * d_{21} \tag{7}$$

We estimate $d_{12} \dots d_{1k}$ data distances by the right segment linking the points d_{11} and d_{21}. Finally, to check (4)…(7) it is necessary to find particulars pixels that can serve as reference and as departure points in and , while working as relative manner.

For the surfaces, the pixels forming S_{befor} and S_{after} are coded in distances to give images of gray level, if we denote by H_{befor} and H_{after} their histograms [3], then for the translations and the rotations we have;

$$S_{befor} = S_{after} \Rightarrow H_{befor} = H_{after} \tag{8}$$

Whereas, for homothety of k ratio $S_{after} = k^2 . S_{before}$.

Having the level i in S_{before}, it will be affected by the level $k.i$ and which repeat $H_{after}(k.i)$ so that

$$H_{after}(k.i) = k * H_{before}(i) \tag{9}$$

Let $H_{before} = \{h(1), h(2), \dots, h(i), \dots, h(n)\}$ and $H_{after} = \{f(1), f(2), \dots, f(k.1), \dots, f(k.2), \dots, f(k.i), \dots, f(k.n)\}$. The two histograms must verify (10) and (11) first their sum;

$$\sum_{j=1..kn} f(j) = k^2 . \sum_{i=1..n} h(i) \tag{10}$$

And their compounds are related by:

$$f(k.i) = k * h(i) \tag{11}$$

Practically, (11) means that the transition from the histogram H_{before} to the histogram H_{after} is performed by a simple linear transformation.

Now, we go to the information reduction step of the image, by doing its code by two vectors, the first is of contour distances and the second is of the surface histograms.

3. Procedure of representations

3.1. Principle of coding

This representation is very delicate if we proceed with theoretical coding of distances as the ultimate erode [4], furthermore it is very computationally prohibitive task. To remedy to these problems we have develop a simple, fast and effective; it consists of creating a mask image M, of gray level formed by concentric circles, where the pixel take as gray level the distances which does separate them from the center as it shown by figure 1 [5] [6]. Later, this mask image is used in the representation as follow:

1. From the binary image that represent the shape, its gravity center is calculated then its contour of thickness 1 pixel.

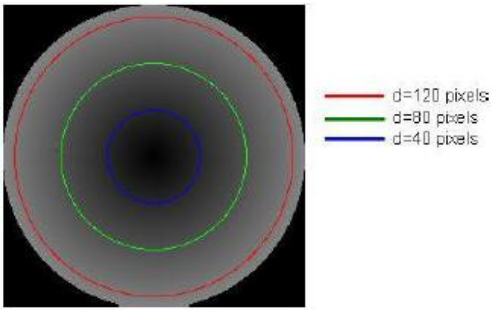

Figure 1 M: Distance mask

2. By simple translation of the mask, we coincide the mask center M with the gravity center of C the shape, the product point by point is done to give two images of gray level, the first contain the contour C where the pixels have as level the distance which separate them from the gravity center of the shape and the second image is of the shape where each pixel is coded in distance.

3.2. Departure pixel choice

For each shape, we have to do a particular and wise choice of the followed departure pixel. For example, for this shape we have chosen the most far pixel regarding the gravity center; which correspond to the maximum of distances. Let E be the set of pixels:

$$E = \{(i,j)/C(i,j) = Max(C)\} \quad (12)$$

If $|E| > 1$, which means several pixels verify this condition, we will see later that we have to pass them one after the other so that we converge to the one which will verify our recognition criteria.

3.3. Loading of representation vectors

According to what is presented in 3.1 and 3.2; for each shape and in relative manner we determine;

1. The reference pixels, which is its gravity center and departure pixel.

2. Using this data and by basic operation of the contour tracking C, the distances met are loaded systematically to the vector distance V defined by (1)(5).

3. Finally the image distances gives the histogram vector H.

4. Recognition Algorithm

Having an object beside the camera, its image is taken then we apply the representation procedure given in paragraph 3. There are two pairs of representation vectors, (V_{model}, H_{model}) for the model and (V_{object}, H_{object}) for the object candidate. The recognition procedure uses the following steps:

Step 1: *Calculation of the two forms ratio:* A global ratio between the two forms is calculated based on the dimension ratio of the two vectors

$$k = \|V_{object}\| / \|V_{model}\|$$

or reverse so to have $k > 1$. In the same time, ratio between the two surfaces is been calculated according to (9) (10)

$$k^2 = S_{object}/S_{model} = \sum_i H_{object}(i) \Big/ \sum_j H_{model}(j)$$

these two ratios should coincide. We have to point out that the surface ratio is more precise than the contour one because this latter is not always smoothed.

Step 2: *Histograms Comparison:* According to (11) the histograms are related and we can switch from one to the other by simple linear transformation. Knowing that the ratio k is calculated form step 1, if $k > 1$ we transform H_{object} if not we transform H_{model} according to (11). The histogram obtained by transformation s compared to the non transformed one. For example for $k > 1$ H_{object} is transformed onto H_{object}^{T} this latter is been compared to H_{model}. Thus, the two histograms have to be equal. If so we follow the process to step 3, if not we stop the comparison and the samples are different.

Step 3: *Contours Comparison:* This step is divided into three elementary stages these are as follow:

3.1: Determinations of departure point and the tracking contour direction in each contour: With respect to the application, a particular point such the most far one or the nearest of the gravity center, is chosen as departure point for the two contours. If V_{object} contains n pixels and V_{model} contains m pixels checking this particularity then we combine n and m pixels, and for each pair of pixel, we try to maximize a topologic similarity function furthermore by combining the incremental angle between positive and negative in each contour, all the possibilities of tracking direction are taken into account. The conditions allowing this maximization are retained. These conditions correspond to departure pixels and the tracking direction in each distance contour.

3.2: Synchronous reading and partial ratios calculation: Two reading operations are simultaneously triggered in C_{object} and C_{model}, their distances are loaded respectively in V_{object} and V_{model}. With an increment $i = i + 1$ for the small vector and k for the great vector $j = round(k * i)$ the ratios of these distances $V_{object}(j), V_{model}(i)$ is calculated then stored in $R(i)$ with R the partial ratios vector.

3.3: Evolution analysis of the partial ratios: Following the loading of vector R, for two identical shapes, the partial ratios have to be close to the global ratio k[7]; then a second decision is taken;

If (the series R is stationary with mean value k) Then

Follow the process to step 4,

Else stop the comparison and

Shapes are different.

EndIf

Step 4: *Refine the comparison:* This step is necessary only for ratios $k \gg 1$ or $k \ll 1$. It consists of doing a linear approximation of the first order to estimate the points V_{object} that have not an image in V_{model}(11). Then, calculate the error as a gap between the reel distances and their estimated counterparts by approximation, these gaps are stored in the vector ε.

If (mean(ε) and variance (ε) are weak) Then

the shapes are similar

of proportionality factor equal to k.

Else shapes are different.

EndIf

5. Experimental results

5.1. The algorithm illustration:

In what follows, we will present the algorithm progress upon two objects showing the obtained results in each step. The two objects are presented by distances as it is shown by figure 2.

Step 1: *Ratios Calculation between the two shapes* The vector V_{object} and V_{model} are calculated $\|V_{object}\| = 330$ and $\|V_{model}\| = 249$, hence the global ratio from the contours is $k = 1.3253$. And from histograms the surfaces are $S_{object} = 7986$ and $S_{model} = 4721$. Using (9) $k^2 = 1.6916$ where the calculated ratio from the surfaces is $\sqrt{k^2} = 1.3006$. the ratios are close then we go to step 2.

Step 2: *Histograms comparison* The images histograms coded in distances are presented in figure 3.a, as we can notice these histograms are different. By using (11), we have kept the model histogram (H_{model} blue) not changed but the object one (H_{object} red) we apply upon it the transformation $H^T_{object}(i) = \frac{1}{1.3006} H_{object}(1.3006 * i)$. In the figure 3.b we have presented the histogram unchanged of the model (H_{model} blue) and the transformed histogram of the object (H^T_{object} red). As we can notice the two histograms coincide perfectly, and to quantify the degree of similarity, we have used the

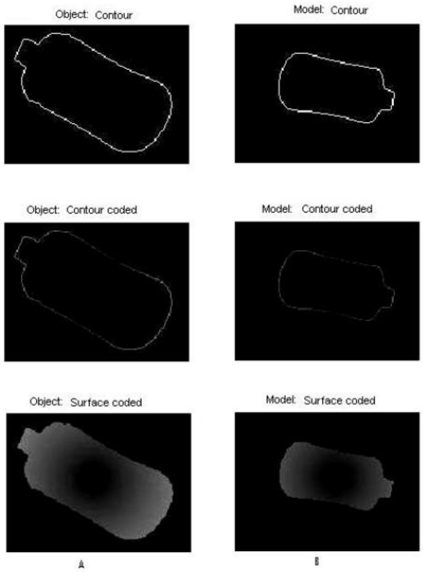

Figure 2 a-object, b-model

criteria of Swain [8]:

$$Dist^{min}_{Swain} = \frac{\sum_{i=1}^{256} \min\left(H^T_{object}(i), H_{model}(i)\right)}{\sum_{i=1}^{256} H_{model}(i)} \quad (13)$$

$$Dist^{max}_{Swain} = \frac{\sum_{i=1}^{256} \max\left(H^T_{object}(i), H_{model}(i)\right)}{\sum_{i=1}^{256} H_{model}(i)} \quad (14)$$

These distances are close to unity for identical histograms, whereas for different histograms $Dist^{min}_{Swain}$ is weak and $Dist^{max}_{Swain}$ is great. In our publication, $Dist^{max}_{Swain} = 1.048$ and $Dist^{min}_{Swain} = 0.9506$, which show effectively that the two histograms coincide perfectly because the two criteria are $\simeq 1$ and we go to step 3.

Step 3: *Contour comparison*

1. The departure pixel as well the direction tracking contour are calculated for the object and for the model they are presented in figure 4.

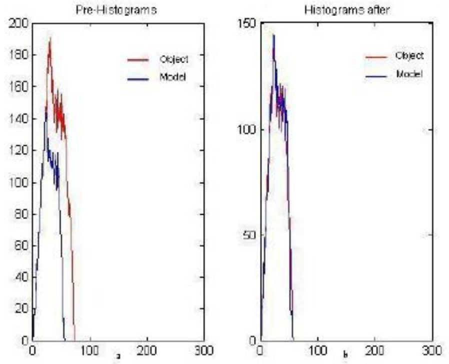

Figure 3 a-pre-histograms, **b**-histograms after

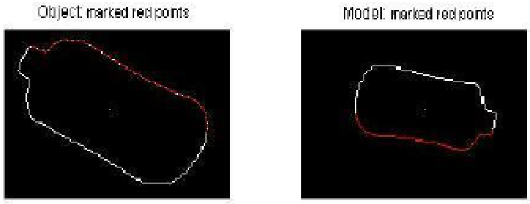

Figure 4 Left: object, Right: model

2. The synchronous reading of the two shapes has allowed the marked red points in figure 4, and the partial ratios vector R is loaded. We notice that still there are white points in V_{object} that not images in V_{model}.

3. The variation of partial ratios stored in R, its average $R_{average} = 1.3018$ its standard deviation $\sigma = 0.0932$ and the accuracy $\frac{|R_{average}-k|}{k} = 0.0009 = 0.09\%$ this confirm that the shapes are related and that one is the image of the other by simple homothety.

Finale decision: *The two shapes are identical and they have a ratio of proportionality of* $k = 1.3006$.

5.2. Recognition of random shapes:

We have constituted two lots of images, the first contains images presenting the same object, only the acquisition conditions are different (camera setting, camera distance, object orientation), and the second constituted of images of different object. Its on these two lots that we have developed a comparative study of the following parameters:

5.2.1. ***Variations of the three ratios***

For the same test, we have calculated three ratios, the one of contour or global k, the one of surfaces and finally the average of partial ratios, obtained in the following of $R_{average}$; results of first lot are presented in figure 5.a and the those of the second lot in figure 5.b.

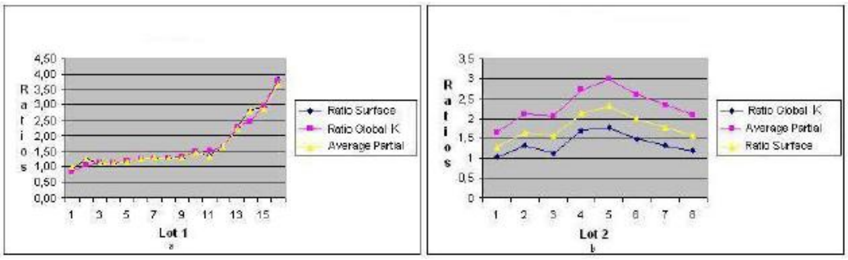

Figure 5 Variations of the three Ratios **a**-lot1 **b**-lot2

5.2.2. *The accuracy variations*

We have marked the accuracy evolution $\frac{|R_{average}-k|}{k}$, the results of the first lot in figure 6.a, the accuracy is less that 5% whereas in the second lot of figure 6.b the accuracy is greater than 27%.

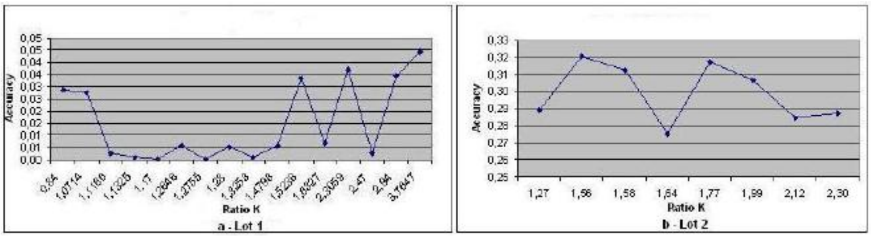

Figure 6 Accuracy variations **a**:lot1 **b**:lot2

5.2.3. *Histograms equality*

The figure 7.a summarizes the variation of $Dist_{Swain}^{min}(13)$, and the $Dist_{Swain}^{max}(14)$ during the different tests of the first lot, and figure 7.b those of the second lot.

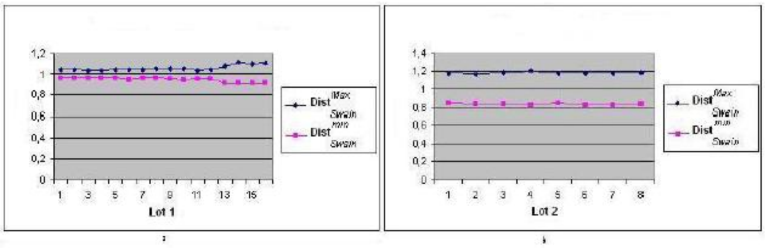

Figure 7 Histograms equality **a**-lot1 **b**-lot2

5.3. Summary

According to our comparative study, the following table summarizes the different parameters.

The parameter	Identical samples	Different samples
Contour ratios, Surface, Average	Equals	Different
Partial ratios Variations	Weak Stationary	Great Not stationary
Accuracy	<5%	>5%
Histograms	Equals	Different
$Dist_{Swain}^{min}$	$\simeq 1$	$\ll 1$
$Dist_{Swain}^{max}$	$\simeq 1$	$\gg 1$

Conclusion

We have presented here, a novel and very simple technique of shapes representation as two vectors, the first is of contour distances and the second is the distances histogram of its surface. The boring step of calculus is accelerated enormously by the use of image mask of distances; which is generated previously outside the recognition procedure.

Thus, the coding is reduced to a product point by point between the mask and the binary image of the shape, and the shape itself becomes a vector of weak dimension, so less memory used, and simple to use, for this reason it is called numerical signature.

As we can notice, our method has two strengths. The first is its simplicity of manipulation, instead of using images it does operate on vectors, so it does manipulate shapes quickly. The second is its working in relative manner; it does manipulate random shapes and without previous setting of the acquisition chain neither for precise positioning.

This last strong point has permitted a great request in industry. In addition with its linear approximation, this technique could compare shapes with different scales, this is very promising as a recognition operations in uncontrolled conditions.

Finally, with determined parameters of recognition, the obtained results are very promising and testify its effectiveness.

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An Estimating the *p*-adic Sizes of Common Zeros of Partial Derivative Polynomials

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Abstract: Let $\underline{x} = (x_1, x_2, \ldots, x_n)$ be a vector in the space Z^n with Z ring of integers and q be a positive integer, f a polynomial in $\underline{x}$ with coefficient in Z. The exponential sum associated with f is defined as $S(f;q) = \sum \underline{x} mod q e^{\frac{2\pi i f(x)}{q}}$, where the sum is taken over a complete set of residues modulo q. The value of $S(f;q)$ depend on the estimate of cardinality $|V|$, the number of elements contained in the set $V = \{\underline{x} mod q | \underline{f_x} = \underline{0} \; mod q\}$ where f_x is the partial derivatives of $\underline{f}$ with respect to $\underline{x}$. To determine the cardinality of V, the p-adic sizes of common zeros of the partial derivative polynomials need to be obtained. In this paper we estimate the *p*-adic sizes of common zeros of partial derivative polynomials of $f(x,y)$ in $Z_p[x,y]$ of degree nine by using Newton polyhedron technique. The degree nine polynomial is of the form $f(x,y) = ax^9 + bx^8y + cx^7y^2 + sx + ty + k$.

Keywords: Exponential sums; cardinality; *p*-adic sizes; Newton polyhedron.

1. Introduction

In our discussion, we use notations the ring of *p*-adic integers Z_p, the completion of algebraic closure of Q_p the field of rational *p*-adic numbers Ω_p and $ord_p x$ is the highest power of p dividing x. It follows that for rational number x and y, $ord_p x = \infty$ if and only if $x = 0$, $ord_p(xy = ord_p x + ord_p y \; and \; ord_p(x + y) \geq min\{ord_p x, ord_p y\}$ with equality if $ord_p x \neq ord_p y$.

Loxton and Vaughan (1985) are the researches who investigate the exponential sums $S(f;q) = \sum_{\underline{x} mod q} exp(2\pi i q)$ where f is a nonlinear polynomial in $Z[\underline{x}]$. They showed that the number of common zeros of the partial derivative polynomials of f with respect to $\underline{x}$ modulo q gives the estimation of $S(f;q)$.

From the works of Loxton and Smith (1982), they found that the *p*-adic sizes of common zeros to partial derivative polynomials associated with f in the neighbourhood of points in the product space $\Omega_p^n, n > 0$, can estimate the cardinality of V. Their result is the estimation of $ord_p(\underline{x} - \xi_i)$ that will lead to a derivation of estimate of $N\left(\underline{f}, p^\alpha\right)$.

The estimations for lower degree two-variable polynomials by using Newton polyhedron technique are found by many researchers such as Mohd. Atan (1986), Chan and Mohd. Atan (1997) who estimates the cardinality $N(f;p^\alpha)$ of the set of solutions to congruence equations modulo a prime power and also Heng and Mohd. Atan (1999). However, results for the polynomials of higher degrees are less complete.

Our approach entails the work developed by Mohd. Atan and Loxton (1986) who presented the *p*-adic Newton polyhedral method of finding the *p*-adic order of polynomials in $\Omega_p[x,y]$ which is an analogue of Newton polygon defined by Koblitz (1977). Sapar and Mohd. Atan (2002) improved the result and then Yap, Sapar and Mohd. Atan (2011) showed that the *p*-adic sizes of common zeros of partial derivative polynomials associated with a cubic form can be found explicitly on the overlapping segment of the indicator diagrams associated with the polynomials by using Newton polyhedron technique.

Our work involves application of the Newton polyhedron technique at the point of intersection in the combination of indicator diagrams to determine explicitly the p-adic sizes of the component (ξ, η) a common root of partial derivative polynomials of $f(x, y)$ in $Z_p[x, y]$ of degree nine.

2. *p*-ADIC Orders of Zeros of A Polynomial

Sapar and Mohd Atan (2002) proved that for every point of intersection of the indicator diagrams, there exist common zeros of both polynomials in $Z_p[x, y]$ whose p-adic orders correspond to point (μ, λ) as mention in Theorem 2.1 below:

Theorem 2.1. Let p be a prime. Suppose f and g are polynomials in $Z_p[x, y]$. Let (μ, λ) be a point of intersection of the indicator diagrams associated with f and g at the vertices or simple points of intersections. Then there are ξ and η in Ω_p satisfying $f\ (\xi, \eta) = g\ (\xi, \eta) = 0$ and $ord_p\xi = \mu,\ ord_p\eta = \lambda$.

Our investigation concentrates on the p-adic sizes of common zeros of partial derivative associated with a polynomial $f(x, y) = ax^9 + bx^8y + cx^7y^2 + sx + ty + k$. First we prove the following lemma.

Lemma 2.1. Let $p > 7$ be a prime, a, b and c in Z_p and λ_1, λ_2 zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$. Let

$$\alpha_1 = \frac{4b + \lambda_1 c}{9a + \lambda_1 b}\ and\ \alpha_2 = \frac{4b + \lambda_2 c}{9a + \lambda_2 b}.$$

If $ord_p b^2 > ord_p ac$, then $ord_p\alpha_i = ord_p(\alpha_1 - \alpha_2) = \frac{1}{2} ord_p\frac{c}{a}$, for $i = 1,2$ and $ord_p(\alpha_1 + \alpha_2) = ord_p\frac{b}{a}$.

Proof. The zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$ are given by

$$\lambda_1 c = \frac{-b \pm \sqrt{252ac - 63b^2}}{2}, for\ i = 1,2.$$

Since $ord_p b^2 > ord_p ac$ and $p > 7$, we have $ord_p\lambda_i c = \frac{1}{2} ord_p ac, i = 1,2$.

Hence, $ord_p\lambda_i c = \frac{1}{2} ord_p ac < ord_p b$. Therefore,

$$ord_p(4b + \lambda c) = ord_p\lambda c = \frac{1}{2} ord_p ac. \tag{2.1}$$

It can be shown that $ord_p\lambda > ord_p a$.
It follows that, $ord_p(9a + \lambda b) = ord_p a$. (2.2)

From (2.1) and (2.2), since $p > 7$, we have

$$ord_p\alpha_i = ord_p\frac{4b + \lambda_i c}{9a + \lambda_i b} = \frac{1}{2} ord_p ac - ord_p a, \qquad i = 1,2.$$

That is $ord_p\alpha_i = \frac{1}{2} ord_p\frac{c}{a}, for\ i = 1,2.$ (2.3)

Clearly,

$$ord_p(\alpha_1 - \alpha_2) = ord_p\frac{(\lambda_1 - \lambda_2)(9ac - 4b^2)}{(9a + \lambda_1 b)(9a + \lambda_2 b)}$$

where $\lambda_1 - \lambda_2 = \frac{\sqrt{252ac-63b^2}}{c}$.

Thus,

$$ord_p(\alpha_1 - \alpha_2) = ord_p\sqrt{252ac - 63b^2} - ord_p c + ord_p(9ac - 4b^2) - ord_p(9a + \lambda_1 b) - ord_p(9a + \lambda_2 b).$$

Since $p > 7, ord_p b^2 > \ ord_p ac$ and by (2.1), (2.2) and (2.3) we have

$$ord_p(\alpha_1 - \alpha_2) = \frac{1}{2} ord_p \frac{c}{a}, for\ i = 1{,}2.$$

It can be shown that

$$ord_p(\alpha_1 + \alpha_2) = ord_p \frac{72ab + 2bc\lambda_1\lambda_2 + (9ac - 4b^2)(\lambda_1 + \lambda_2)}{(9a + \lambda_1 b)(9a + \lambda_2 b)} \tag{2.4}$$

where $\lambda_1\lambda_2 = \frac{64b^2-252ac}{4c^2}$ and $\lambda_1 + \lambda_2 = -\frac{b}{a}$.

Since $p > 7, ord_p b^2 > \ ord_p ac$ and $ord_p(9a + \lambda_i b) = ord_p a, i = 1{,}2$ and simplifying (2.4) we have

$$ord_p(\alpha_1 + \alpha_2) = ord_p \frac{b}{a}$$

as asserted.

Throughout the following discussion,

$$\alpha_1 = \frac{4b + \lambda_1 c}{9a + \lambda_1 b}\ and\ \alpha_2 = \frac{4b + \lambda_2 c}{9a + \lambda_2 b} \tag{2.5}$$

with λ_1, λ_2 zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac. \alpha_1 \neq \alpha_2$ since $\lambda_1 \neq \lambda_2$.

Lemma 2.2. Suppose (U, V) in Ω_p^2. Let $p > 7$ be a prime, a, b and c coefficient of $\alpha_1$and α_2 as in (2.5) Z_p. If $ord_p b^2 > ord_p ac$, then $ord_p(\alpha_1 V - \alpha_2 U) = ord_p\left[7b(U - V) + \sqrt{252ac - 63b^2}(U + V)\right] - ord_p a.$

Proof.

$$\begin{aligned} ord_p(\alpha_1 V - \alpha_2 U) &= ord_p \left(\frac{4b + \lambda_1 c}{9a + \lambda_1 b} V - \frac{4b + \lambda_2 c}{9a + \lambda_2 b} U\right) \\ &= ord_p[(4b + \lambda_1 c)(9a + \lambda_2 b)V - (4b + \lambda_2 c)(9a + \lambda_1 b)U] - ord_p(9a + \lambda_1 b) - ord_p(9a + \lambda_2 b). \end{aligned} \tag{2.6}$$

Now, let λ_1 and λ_2 be the zeros of $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$ are of the form

$$\lambda_1 = \frac{-b + \sqrt{252ac - 63b^2}}{2c} \text{ and } \lambda_2 = \frac{-b - \sqrt{252ac - 63b^2}}{2c}.$$

From (2.6), we have

$$(4b + \lambda_1 c)(9a + \lambda_2 b)V - (4b + \lambda_2 c)(9a + \lambda_1 b)U = \left(\frac{9ac-4b^2}{2c}\right)\left[7b(U - V) + \sqrt{252ac - 63b^2}(U + V)\right].$$

Therefore, from (2.6) we have

$$ord_p(\alpha_1 V - \alpha_2 U) = ord_p\left(\frac{9ac - 4b^2}{2c}\right)\left[7b(U - V) + \sqrt{252ac - 63b^2}(U + V)\right] - ord_p(9a + \lambda_1 b) - ord_p(9a + \lambda_2 b)$$

Since $ord_p b^2 > ord_p ac$, and by proof of Lemma 2.1, $ord_p(9a + \lambda_i b) = ord_p a$ for $i = 1{,}2$, we obtain

$$ord_p(\alpha_1 V - \alpha_2 U) = ord_p\left[7b(U-V) + \sqrt{252ac - 63b^2}(U+V)\right] - ord_p a$$

as asserted.

From the above result, it is clear that to ascertain the p-adic sizes of $ord_p(\alpha_1 V - \alpha_2 U)$ we need to examine the p-adic size of $\left[7b(U-V) + \sqrt{252ac - 63b^2}(U+V)\right]$. To do this, the sizes of each quantity in the expression should be considered. This is done in the proof of the following assertion.

Lemma 2.3. Suppose(x, y) in Ω_p^2 and $U = x^4 + \alpha_1 x^3 y, V = x^4 + \alpha_2 x^3 y$ where α_1 and α_2 as in (2.5). Let $p > 7$ be a prime, a, b and c coefficient of $\alpha_1$and α_2 $Z_p$and $ord_p b^2 > ord_p ac$. Then $ord_p x \geq \frac{1}{4}W$ and $ord_p y \geq \frac{1}{4}\big[W -$ *12ordpcb6a7* or *ordpy≥14W−12ordpcb6a7−3ε* in an exceptional case with *W=minordpV,ordpU* and some *ε≥0* which can be specified explicitly.

Proof. From $U = x^4 + \alpha_1 x^3 y$ and $V = x^4 + \alpha_2 x^3 y$, we have

$$x = \left(\frac{\alpha_1 V - \alpha_2 U}{\alpha_1 - \alpha_2}\right)^{\frac{1}{4}} \text{ and } y = \frac{U - V}{(\alpha_1 - \alpha_2)x^3}.$$

Thus,$ord_p x = \frac{1}{4} ord_p(\alpha_1 V - \alpha_2 U) - \frac{1}{4} ord_p(\alpha_1 - \alpha_2)$ (2.7)

and $ord_p y = ord_p(U - V) - ord_p(\alpha_1 - \alpha_2) - ord_p x^3$. (2.8)

By (2.7), Lemmas 2.1 and 2.2, we have

$$ord_p x = \frac{1}{4} ord_p\left[7b(U-V) + \sqrt{252ac - 63b^2}(U+V)\right] - \frac{1}{8} ord_p ac.$$

Now, we have to consider two cases.

<u>Case 1:</u> $\left\{ord_p 7b(U-V) \neq ord_p\sqrt{252ac - 63b^2}(U+V)\right\}$

(i) Suppose $min\left\{ord_p 7b(U-V), ord_p\sqrt{252ac - 63b^2}(U+V)\right\} = ord_p\sqrt{252ac - 63b^2}(U+V)$

It follow that, $ord_p x = \frac{1}{4} ord_p\sqrt{252ac - 63b^2}(U+V) - \frac{1}{8} ord_p ac$.

Since $p > 7$ and $ord_p b^2 > ord_p ac$, we have

$$ord_p x = \frac{1}{4} ord_p(U+V) \tag{2.9}$$

It follow that, $ord_p x \geq \frac{1}{4}W$.

From the definition of U and V,

$$ord_p(U+V) = ord_p(2x^4 + (\alpha_1 + \alpha_2)x^3 y).$$

From (2.9),

$$ord_p x^4 = ord_p(U+V).$$

Thus,

$$ord_p x \leq ord_p(\alpha_1 + \alpha_2)y.$$

Hence from (2.8), we have

$$ord_p y \geq ord_p(U-V) - ord_p(\alpha_1 - \alpha_2) - ord_p(\alpha_1 + \alpha_2)^3 - 3ord_p y$$

and from the proof of Lemma 2.1 and simplify it, we have

$$ord_p y \geq ord_p(U-V) - \frac{1}{2}\left[ord_p \frac{c}{a} ord_p \frac{b^3}{a^3}\right]$$

by Lemma 2.1, we have

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right].$$

Hence, in this case,

$$ord_p x \geq \frac{1}{4} W \text{ and } ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right].$$

(ii) Suppose $min\left\{ord_p 7b(U-V), ord_p\sqrt{252ac - 63b^2}(U+V)\right\} = ord_p 7b(U-V)$ (2.10)

We have

$$ord_p x = \frac{1}{4} ord_p 7b(U-V) - \frac{1}{8} ord_p ac = \frac{1}{4} ord_p(U-V) + \frac{1}{8}\left(ord_p b^2 - ord_p ac\right). \quad (2.11)$$

Since $ord_p b^2 > ord_p ac$, we have

$$ord_p x^4 \geq ord_p(U-V).$$

That is, $ord_p x = \frac{1}{4} W$.

By (2.10) and (2.11),

$$ord_p x \leq \frac{1}{4} ord_p\sqrt{252ac - 63b^2}(U+V) - \frac{1}{8} ord_p ac = \frac{1}{8} ord_p ac + \frac{1}{4} ord_p(U+V) - -\frac{1}{8} ord_p ac.$$

That is, $ord_p x \leq \frac{1}{4} ord_p(U+V)$.

Now $(U+V) = 2x^4 + (\alpha_1 + \alpha_2)x^3 y$. Thus,

$$ord_p x \leq ord_p(2x + (\alpha_1 + \alpha_2)y).$$

It follows that,

$$ord_p x \leq ord_p(\alpha_1 + \alpha_2)y.$$

By (2.8), and the same argument as in (i) we have,

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right].$$

<u>Case2:</u> $\left\{ord_p 7b(U-V) = ord_p\sqrt{252ac - 63b^2}(U+V)\right\}$.

We have

$$ord_p x = \frac{1}{4}\left[ord_p 7b(U-V) + ord_p\sqrt{252ac-63b^2}(U+V)\right] - \frac{1}{8}ord_p ac$$
$$\geq \frac{1}{4}min\left\{ord_p 7b(U-V), ord_p\sqrt{252ac-63b^2}(U+V)\right\} - \frac{1}{8}ord_p ac.$$

Since $ord_p 7b(U-V) = ord_p\sqrt{252ac-63b^2}(U+V)$ and $p > 7$,

$$ord_p x \geq \frac{1}{4}ord_p\sqrt{252ac-63b^2}(U+V) - \frac{1}{8}ord_p ac.$$

Since $p > 7$ and $ord_p b^2 > ord_p ac$, we have

$$ord_p x \geq \frac{1}{4}ord_p(U+V) + \frac{1}{8}\left(ord_p ac - ord_p ac\right).$$

Therefore, $ord_p x \geq \frac{1}{4}ord_p(U+V)$.

It follows that, $ord_p x \geq \frac{1}{4}W$.

From (2.7) and (2.8), we obtain

$$ord_p y = ord_p(U-V) - ord_p(\alpha_1-\alpha_2) - 3\left[\frac{1}{4}ord_p(\alpha_1 V - \alpha_2 U) - \frac{1}{4}ord_p(\alpha_1-\alpha_2)\right].$$

By Lemmas 2.1 and 2.2, we obtain

$$ord_p y = ord_p(U-V) - \frac{1}{8}ord_p\frac{c}{a^7} - \frac{3}{4}ord_p\left[7b(U-V) + \sqrt{252ac-63b^2}(U+V)\right]. \tag{2.12}$$

Let, $\beta = ord_p 7b(U-V) = ord_p\sqrt{252ac-63b^2}(U+V)$. (2.13)

Then, there exist k and l such that,

$7b(U-V) = p^\beta k$ with $ord_p k = 0$ and $\sqrt{252ac-63b^2} = p^\beta l$ with $ord_p l = 0$.

From (2.13), $ord_p(U-V) = \beta - ord_p b$. Hence from (2.12), we have

$$ord_p y = \frac{1}{4}\beta - ord_p b - \frac{1}{8}ord_p\frac{c}{a^7} - \frac{3}{4}ord_p(k+l).$$

Let $\varepsilon = ord_p(k+l)$, then

$$ord_p y = \frac{1}{4}ord_p(U-V) - \frac{1}{8}ord_p\frac{cb^6}{a^7} - \frac{3}{4}\varepsilon.$$

It follows that,

$$ord_p y \geq \frac{1}{4}W - \frac{1}{8}ord_p\frac{cb^6}{a^7} - \frac{3}{4}\varepsilon.$$

Hence, we have

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2}ord_p\frac{cb^6}{a^7} - 3\varepsilon\right]$$

with $W = \left\{ord_p V, ord_p U\right\}$ and $\varepsilon ord_p(k+l)$.

Therefore, $ord_p x \geq \frac{1}{4}W$ and $ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2}ord_p \frac{cb^6}{a^7}\right]$ or $ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2}ord_p \frac{cb^6}{a^7} - 3\varepsilon\right]$

with $W = \{ord_p V, ord_p U\}$ and $\varepsilon \geq 0$ as asserted.

The following lemma gives explicit estimates of the components x, y in U and V in terms of p-adic sizes of integers in Z_p where U and V as in Lemma 2.3. the proof utilizes the result obtained above.

Lemma 2.4. Suppose(x, y) in Ω_p^2 and $U = x^4 + \alpha_1 x^3 y, V = x^4 + \alpha_2 x^3 y$ where α_1 and α_2 as in (2.5). Let $p > 7$ be a prime, a, b, c, s and t in Z_p $ord_p b^2 > ord_p ac, \delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p s, ord_p t \geq \delta$. If $ord_p U = \frac{1}{2}ord_p \frac{s+\lambda_1 t}{9a+\lambda_1 b}$ and $ord_p V = \frac{1}{2}ord_p \frac{s+\lambda_2 t}{9a+\lambda_2 b}$ then $ord_p x \geq \frac{1}{8}(\alpha - \delta)$ and $ord_p y \geq \frac{1}{8}(\alpha - \delta)$ or $ord_p y \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ for some $\varepsilon \geq 0$.

Proof. Since $U = x^4 + \alpha_1 x^3 y, V = x^4 + \alpha_2 x^3 y$ and $ord_p b^2 > ord_p ac$, we have from Lemma 2.3

$$ord_p x \geq \frac{1}{4}W \tag{2.14}$$

where $W = min\{ord_p U, ord_p V\}$.

Now,

$$ord_p U = \frac{1}{2}ord_p \frac{s + \lambda_1 t}{9a + \lambda_1 b} \text{ and } ord_p V = \frac{1}{2}ord_p \frac{s + \lambda_2 t}{9a + \lambda_2 b}.$$

It follows from (2.14) that

$$ord_p x \geq \frac{1}{8}ord_p \frac{s + \lambda_i t}{9a + \lambda_i b}, \;\; i = 1 \text{ or } 2.$$

By proof of Lemma 2.1, $ord_p(9a + \lambda_i b) = ord_p a$ for $i = 1, 2$. As such

$$ord_p x \geq \frac{1}{8}\left[ord_p(s + \lambda_i t) - ord_p a\right] \tag{2.15}$$

If $min\{ord_p s, ord_p \lambda_i t\} = ord_p s, i = 1, 2$ then

$$ord_p x \geq \frac{1}{8}\left(ord_p s - ord_p a\right).$$

By the hypothesis, we obtain

$$ord_p x \geq \frac{1}{8}(\alpha - \delta).$$

Now, if $min\{ord_p s, ord_p \lambda_i t\} = ord_p \lambda_i t, i = 1, 2$ then

$$ord_p x \geq \frac{1}{8}\left[ord_p \lambda_i t - ord_p a\right].$$

Since $ord_p a \leq ord_p \lambda_i b, i = 1, 2$ it follows that

$$ord_p x \geq \frac{1}{8}\left[ord_p t - ord_p b\right].$$

By the hypothesis, we obtain

$$ord_p x \geq \frac{1}{8}(\alpha - \delta).$$

By Lemma 2.3, we have

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right] \text{ or } ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7} - 3\varepsilon\right] \tag{2.16}$$

for some $\varepsilon \geq 0$ where $W = min\{ord_p U, ord_p V\}$.

For the first inequality we have from (2.16),

$$ord_p y \geq \frac{1}{4}\left[\frac{1}{2} ord_p \left(\frac{s + \lambda_i t}{9a + \lambda_i b}\right) - \frac{1}{2} ord_p \frac{cb^6}{a^7}\right], i = 1, 2.$$

Since $ord_p(9a + \lambda_i b) = ord_p a$ for $i = 1, 2$,

$$ord_p y \geq \frac{1}{8}\left[ord_p(s + \lambda_i t) + ord_p a^6 - ord_p b^6 c\right] \tag{2.17}$$

Since $ord_p b^2 > ord_p ac$, we have

$$ord_p y \geq \frac{1}{8}\left[ord_p(s + \lambda_i t) + ord_p a\right].$$

By using the same method as equation (2.15), we have

$$ord_p y \geq \frac{1}{8}(\alpha - \delta).$$

Now, we consider the second inequality,

$$ord_p y \geq \frac{1}{4}\left[W - \frac{1}{2} ord_p \frac{cb^6}{a^7} - 3\varepsilon_0\right]$$

with $W = min\{ord_p U, ord_p V\}$ and for some $\varepsilon_0 \geq 0$.

By the same argument for the first inequality not involving ε_0, we let $\varepsilon = 3\varepsilon_0$ and we will arrive at

$$ord_p y \geq \frac{1}{8}(\alpha - \delta - \varepsilon).$$

Therefore, $ord_p x \geq \frac{1}{8}(\alpha - \delta)$ and $ord_p y \geq \frac{1}{8}(\alpha - \delta)$ or $ord_p y \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$

as asserted.

The next theorem will gives the p-adic sizes of common zeros of partial derivative polynomials associated with a polynomial $f(x, y)$ in $Z_p[x, y]$, in terms of the coefficients of its dominant terms.

Theorem 2.2. Let $f(x, y) = ax^9 + bx^8 y + cx^7 y^2 + sx + ty + k$ be a polynomial in $Z_p[x, y]$ with $p > 7$. Let $\alpha > 0, \delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p b^2 > ord_p ac$. İf $ord_p f_x(0,0)$, $ord_p f_y(0,0) \geq \alpha > \delta$ then there exists (ξ, η) such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $ord_p \xi \geq \frac{1}{8}(\alpha - \delta)$, $ord_p \eta \geq \frac{1}{8}(\alpha - \delta)$ or in an exceptional case $ord_p \eta \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ with a certain $\varepsilon \geq 0$.

Proof. Let $g = f_x$ and $h = f_y$ and λ be a constant where, $g = f_x = 9ax^8 + 8bx^7 y + 7cx^6 y^2 + s$ and $h = f_y = bx^8 + 2cx^7 y + t$.

Then,

$(g+\lambda h)(x,y) = (9a+\lambda b)x^8 + (8b+2\lambda c)x^7y + 7cx^6y^2 + s + \lambda t$. That is

$$\frac{(g+\lambda h)(x,y)}{9a+\lambda b} = x^8 + \left(\frac{8b+2\lambda c}{9a+\lambda b}\right)x^7y + \left(\frac{7c}{9a+\lambda b}\right)x^6y^2 + \frac{s+\lambda t}{9a+\lambda b}. \tag{2.18}$$

By completing the square in (2.18), we have

$$\frac{(g+\lambda h)(x,y)}{9a+\lambda b} = \left(x^4 + \frac{4b+\lambda c}{9a+\lambda b}x^3y\right)^2 + \frac{s+\lambda t}{9a+\lambda b}. \tag{2.19}$$

with

$$\frac{7c}{9a+\lambda b} - \left(\frac{4b+\lambda c}{9a+\lambda b}\right)^2 = 0 \tag{2.20}$$

That is, $c^2\lambda^2 + bc\lambda + 16b^2 - 63ac = 0$.

From the equation (2.20) above, we have

$$\lambda_1 = \frac{-b+\sqrt{252ac-63b^2}}{2c} \text{ and } \lambda_2 = \frac{-b-\sqrt{252ac-63b^2}}{2c}.$$

Let the above λ_1, λ_2 be the zeros of the equation (2.20) whose expressions are given in Lemma 2.1. $\lambda_1 \neq \lambda_2$, since $ord_p b^2 > ord_p ac$ implies $b^2 \neq 4ac$.

Now let

$$U = x^4 + \frac{4b+\lambda_1 c}{9a+\lambda_1 b}x^3y \tag{2.21}$$

$$V = x^4 + \frac{4b+\lambda_2 c}{9a+\lambda_2 b}x^3y \tag{2.22}$$

$$F(U,V) = (g+\lambda_1 h)(x,y) \tag{2.23}$$

and

$$F(U,V) = (g+\lambda_2 h)(x,y) \tag{2.24}$$

Substitution of U and V in (2.19), for $i = 1, 2$, we have polynomials in (U,V),

$$F(U,V) = (9a+\lambda_1 b)U^2 + s + \lambda_1 t \tag{2.25}$$

$$F(U,V) = (9a+\lambda_2 b)U^2 + s + \lambda_2 t \tag{2.26}$$

The combination of the indicator diagrams associated with the Newton polyhedron of (2.25) and (2.26) is shown in figure below

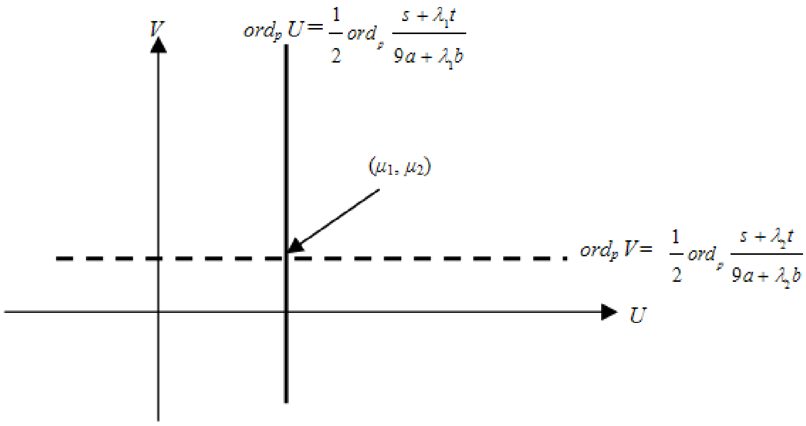

Figure 2.2.1. The indicator diagrams of $F(U,V) = (9a+\lambda_1 b)U^2 + s + \lambda_1 t$ (bold line) and $F(U,V) = (9a+\lambda_2 b)U^2 + s + \lambda_2 t$ (broken line)

From Figure 2.2.1 and by Theorem 2.1, there exists $(\widehat{U}, \widehat{V})$ in Ω_p^2 such that $F(\widehat{U}, \widehat{V}) = 0$, $F(\widehat{U}, \widehat{V}) = 0$ and $ord_p \widehat{U} = \mu_1$, $ord_p \widehat{V} = \mu_2$ with $\mu_1 = \frac{1}{2} ord_p \frac{s+\lambda_1 t}{9a+\lambda_1 b}$ and $\mu_2 = \frac{1}{2} ord_p \frac{s+\lambda_2 t}{9a+\lambda_2 b}$.

Suppose $U = \widehat{U}$ and $V = \widehat{V}$ in (2.21) and (2.22). Thus, there exists (x_0, y_0) in Ω_p^2 such that

$$\widehat{U} = x_0^4 + \alpha_1 x_0^3 y_0 \tag{2.27}$$
$$\widehat{V} = x_0^4 + \alpha_2 x_0^3 y_0 \tag{2.28}$$

with $\alpha_1 = \frac{4b+\lambda_1 c}{9a+\lambda_1 b}$ and $\alpha_2 = \frac{4b+\lambda_2 c}{9a+\lambda_2 b}$, λ_1, λ_2 the zeros $k(\lambda) = c^2\lambda^2 + bc\lambda + 16b^2 - 63ac$. $\alpha_1 \neq \alpha_2$ since $\lambda \neq \lambda$.

By solving (2.27) and (2.28) simultaneously, we have

$$x_0 = \left(\frac{\alpha_1 \widehat{V} - \alpha_2 \widehat{U}}{\alpha_1 - \alpha_2}\right)^{\frac{1}{4}} \ and \ y_0 = \frac{\widehat{U} - \widehat{V}}{(\alpha_1 - \alpha_2)x_0^3}.$$

That is,

$$ord_p x_0 = \frac{1}{4} ord_p(\alpha_1 \widehat{V} - \alpha_2 \widehat{U}) - \frac{1}{4} ord_p(\alpha_1 - \alpha_2) \tag{2.7}$$

and

$$ord_p y_0 = ord_p(\widehat{V} - \widehat{U}) - ord_p(\alpha_1 - \alpha_2) - ord_p x_0^3.$$

From Lemma 2.4, we have

$$ord_p x_0 \geq \frac{1}{8}(\alpha - \delta), ord_p y_0 \geq \frac{1}{8}(\alpha - \delta) \text{ or } ord_p y_0 \geq \frac{1}{8}(\alpha - \delta - \varepsilon) \text{ for some } \varepsilon \geq 0.$$

Let $x_0 = \xi$ and $y_0 = \eta$. Since $F(\widehat{U}, \widehat{V}) = 0$ and $G(\widehat{U}, \widehat{V}) = 0$, by back substitution in (2.23) and (2.24) we would have $g(\xi, \eta) = f_x(\xi, \eta) = 0$ and $h(\xi, \eta) = f_y(\xi, \eta) = 0$. Thus, $ord_p \xi \geq \frac{1}{8}(\alpha - \delta)$, $ord_p \eta \geq \frac{1}{8}(\alpha - \delta)$ or $ord_p \eta \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ where (ξ, η) is a common zero of f_x and f_y $\delta = max\{ord_p a, ord_p b, ord_p c\}$, for some $\varepsilon \geq 0$.

3. Conclusion

From this project, we found that if p is a prime, $p > 7, f(x,y) = ax^9 + bx^8 y + cx^7 y^2 + sx + ty + k$ with all coefficients in Z_p such that for $\alpha > 0, \delta = max\{ord_p a, ord_p b, ord_p c\}$ and $ord_p b^2 > ord_p ac$ if $ord_p f_x(0,0)$,

$ord_p f_y(0,0) \geq \alpha > \delta$ then there exists (ξ, η) such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and $ord_p \xi \geq \frac{1}{8}(\alpha - \delta)$, $ord_p \eta \geq \frac{1}{8}(\alpha - \delta)$ or in an exceptional case $ord_p \eta \geq \frac{1}{8}(\alpha - \delta - \varepsilon)$ with a certain $\varepsilon \geq 0$.

The *p*-adic sizes of common zeros that we obtained in this project can be used to find the cardinality $|V|$ and through that we can solve the exponential sums $S(f;q) = \sum \underline{x} mod q exp(2\pi i/q)$ that depended from estimate of cardinality. Therefore, we also suggest that by using the same technique as in this project, the *p*-adic sizes of common zeros of partial derivative polynomials associated with much higher degree two-variable polynomials also can be found.

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CONTINUED FRACTION EXPANSIONS OF SOME FUNCTIONS OF POSITIVE DEFINITE MATRICES

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Abstract: In this paper we recall some results of matrix functions with eal cofficients. The aim of this paper is to provide some properties and results of continued fractions with matrix arguments. Then we give continued fractions expansions of some inverse of hyperbolic and circular functions $\arcsin(\mathrm{A})$, $\operatorname{arcsinh}(\mathrm{A})$, $\arccos \mathrm{A}$ and $\operatorname{arcch}(\mathrm{A})$ where A is a positive definite matrix.

Keywords: Continued fraction expansion, positive definite matrix, function of matrices

1. Introduction and mativation

Over the last two hundred years, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms.

Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications (see [5],[7], [11]). The real case is relatively well studied in the literature. However, in contrast to the theoretical importance, one can find in mathematical literature only a few results on the continued fractions with matrices arguments. The main difficulty arises from the fact that the algebra of square matrices is not commutative.

For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.

2. Preleminary and notations

Matrix functions play a widespreased role in science and engineering, with applications areas ranging from nuclear magnetic resonance [2]. So for any scalar polynomial $p(z) = \sum_{i=0}^{k} \alpha_i z^i$ gives rise to a matrix polynomial with scalar coefficients by simply substituting A^i ve z^i:

$$P(A) = \sum_{i=0}^{k} \alpha_i A^i$$

More generally, for function f with a series representation on an open disk containing the eigenvalues of A, we are able to define the matrix function $f(A)$ via the Taylor series for f [4].

Alternatively, given a function $f(z)$ that is analytic inside on a closed contour Γ which encircles the eigenvalues of $A, f(A)$ can be defined, by analogy with Cauchy's integral theorem, by

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(zI - A)^{-1}dz$$

The definition is known as the matrix version of Cauchy's integral theorem. We now mention an important result of matrix functions.

Lemma 2.1 *(i) If two matrices $A \in M_m$ and $B \in M_m$ are similar, with*

$$A = XBX^{-1}$$

Then the matrices $f(A)$ and $f(B)$ are also similar, with

$$f(A) = Xf(B)X^{-1}$$

(ii) If $B \in M_m$ is a block diagonal matrix

$$f(B) = diag(f(B_1), f(B_2), \dots, f(B_r))$$

Proof. For $A = XBX^{-1}$ we have $A^k = ZB^kZ^{-1}$. Hence for every polynomial p it follow that

$$p(A) = Xp(B)X^{-1}$$

Therefore if either one of $p(A)$ or $p(B)$ equals zero then so does the other, implying that A and B share the same minimal polynomial. From definition there exists an interpolating polynomial $r(Z)$ such that

$$f(A) = r(A),\ \ f(B) = r(B)$$

and since for every polynomial we have $p(A) = Zp(B)Z^{-1}$, the result follows.

(ii) We deduce it from (i).

Let $A \in M_m, A$ is said to be positive semidefinite (resp. positive definite) if A is symmetric and

$$\forall x \in \mathbb{R}^m, < Ax, x > \geq 0\ (resp. \forall x \in \mathbb{R}^m, x \neq 0 < Ax, x > > 0)$$

where <.,.> denotes the standard scalar product of $\mathbb{R}^m$.

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if A and B are two symmetric matrices, we write $A \leq B\ if\ B - A$ is positive semidefinite.

Henceforth, whenever we say that $A \in M_m$ is positive semidefinite (or positive definite), it will be assumed that A is symmetric.

For any matrices $A, B \in M_m$ with B invertible, we write $\frac{A}{B} = B^{-1}A$. It is easy to verify that for any invertible matrix X we have

$$\frac{A}{B} = \frac{XA}{XB} \neq \frac{AX}{BX}$$

Definition 2.2 Let $(A_n)\ n \geq 0$ and $(B_n)\ n \geq 0$ be two sequences of matrices in M_m. We define the sequences $(P_n)\ n \geq -1$ and $(Q_n)\ n \geq -1$ by

$$\begin{cases} P_{-1} = I,\ P_0 = A_0 \\ Q_{-1} = 0,\ Q_0 = I \end{cases} \quad and \quad \begin{cases} P_n = A_nP_{n-1} + B_nP_{n-2} \\ Q_n = A_nQ_{n-1} + B_nQ_{n-2} \end{cases} \quad n \geq 1 \tag{2.1}$$

The matrix P_n/Q_n is called the n^{th} convergent of $K(B_n/A_n)$, the fraction $\frac{B_n}{A_n}$ is called its n^{th} partial quotient. The proof of the next proposition is elementary and we left it to the reader.

Propotion 2.3. *For any two matrices C and D with C invertible, we have*

$$C\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{n} \qquad D = \left[CA_0D;\frac{B_1D}{A_1C^{-1}},\frac{B_2C^{-1}}{A_2},\frac{B_k}{A_k}\right]_{k=3}^{n}$$

Definition 2.4. Let $(A_n),(B_n),(C_n)$ *and* (D_n) be four sequences of matrices. We say that the continued fractions $K(B_n/A_n)$ and $K(D_n/C_n)$ are equivalent if we have $F_n = G_n$ for all $n \geq 1$, where F_n and G_n are the n^{th} convergents of $K(B_n/A_n)$ and $K(D_n/C_n)$ respectively.

In order to simplify the statements on some partial quotients of continued fractions with matrices arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.5. *(see [10]) Let* $\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ *be a given continued fraction.*

Then

$$\frac{P_n}{Q_n} = \left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0;\frac{X_kB_kX_{k-2}^{-1}}{X_kA_kX_{k-2}^{-1}}\right]_{k=1}^{n},$$

where $X_{-1} = X_0 = I$ *and* $X_1, X_2, \dots, X_n$ *are arbitrary invertible matrices.*

We also recall the following proposition in real cace.

Proposition 2.6. *Let* (r_n) be a non-zero sequence of real numbers. The following continued fractions

$$\left[a_0;\frac{b_1}{a_1},\frac{b_2}{a_2},\frac{b_3}{a_2},\dots\right] \quad and \quad \left[a_0;\frac{r_1b_1}{r_1a_1},\frac{r_2r_1b_2}{r_2a_2},\frac{r_3r_2b_3}{r_3a_2},\dots\right]$$

are equivalent.

We end this section by introducing some topological notions of continued fractions with matrix arguments. We provide M_m with the topology induced by the following classical norm:

$$\forall A \in M_{m,}\ \|A\| = sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = sup_{\|x\|=1}\|Ax\|$$

Definition 2.7. Let $\{A_n\}$ be a sequence of matrices in M_m. We say that $\{A_n\}$ converges in M_m if there exists a matrix $A \in M_m$ such that $\|A_n - A\|$ tends to 0 when n tens to $+\infty$. In this case we write ,$\lim_{n\to\infty} A_n = A$.

The continued fraction $\left[A_0;\frac{B_k}{A_k}\right]_{k=1}^{+\infty}$ is said to be convergent in M_m if the sequence $\{P_n/Q_n\}_n = \{Q_n^{-1}P_n\}_n$ converges in M_m in the sense that there exists a matrix $F \in M_m$ such that $\lim_{n\to+\infty}\|F_n - F\| = 0$

3. Man Result

Let $A \in M_m$ be a positive definite matrix. Our aim in this section is to give a continued fraction expansion of arcsin, arcsh(A), arccos(A) and arcch(A). For simplicity, we start with the real case and we begin by recalling Laguerre's continued fraction in the following lemma.

Lemma 3.11 *(see [3]) Let x be a real number such that* $0 < x < 1$. *Then there holds*

$$\frac{\arcsin(x)}{(1-x^2)^{1/2}} = \left[0;\frac{x}{1},\frac{-(2k-2)(2k)x^2}{-(4k-1)}, \qquad \frac{-(2k-2)(2k)x^2}{-(4k+1)}\right]_{k=1}^{+\infty} \tag{3.1}$$

Now we establish a main theorem which, is a matrix version of the previous lemma.

Theorem 3.2. *Let* $A \in M_m$ *be a positive definite matrix such that* $\|A\| < 1$. *Then a continued fraction expansion of arcsin(A) is*

$$\text{arcsin}\,(A) = \left[0; \frac{A(I-A^2)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{-(4k-1)I}, \frac{(2k-1)(2k)A^2}{-(4k+1)I}\right]_{k=1}^{+\infty} \tag{3.2}$$

Since $arcsh(A) = -iarcsin(A)$, by vertu of proposition 2.6 and the previous theorem, we have the next result.

Corollary 3.3. *A continued fraction expansion of arcsh(A) is given by*

$$\text{arcsh}\,(A) = \left[0; \frac{A(I-A^2)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{-(4k-1)I}, \quad \frac{(2k-1)(2k)A^2}{-(4k+1)I}\right]_{k=1}^{+\infty} \tag{3.3}$$

Proof. Let A ∈ $\mathrm{M_m}$ be a positive definite matrix Then there exists an invertible matrix X such that $A = XDX^{-1}$, where $D = diag(\lambda_1, \lambda_2, \dots, \lambda_m)$ and $\lambda_i > 0$.

As the function $z \to \text{arcsin}\,(z)$ is analytic in the open halfplane $\{z \in \mathbb{C}/Re(z) > 0\}$, then

$$\text{arcsin}(A) = X(\text{arcsin}(D))X^{-1} = X\,diag(\text{arcsin}(\lambda_1), \text{arcsin}(\lambda_2), \dots, \text{arcsin}(\lambda_m))X^{-1}.$$

Let us define the sequences $\{P_n\}$ and $\{Q_n\}$ by:

$$\begin{cases} P_{-1} = I, P_0 = 0, P_1 = D \\ Q_{-1} = O, Q_0 = I, Q_1 = I \end{cases}$$

and for $k \geq 1$,

$$\begin{cases} P_{2k} = -(4k-1)P_{2k-1} + (2k-1)(2k)D^2 P_{2k-2}, \\ Q_{2k} = -(4k-1)Q_{2k-1} + (2k-1)(2k)D^2 Q_{2k-2}, \end{cases}$$

$$\begin{cases} P_{2k+1} = -(4k-1)P_{2k} + (2k-1)(2k)D^2 P_{2k-1}, \\ Q_{2k+1} = -(4k-1)Q_{2k} + (2k-1)(2k)D^2 Q_{2k-1}, \end{cases}$$

We see that P_n and Q_n are diagonal matrices, by setting $p_n = diag(p_n^1, p_n^2, \dots, p_n^m)$ and $q_n = diag(q_n^1, q_n^2, \dots, q_n^m)$, we obtain for each $i, 1 \leq i \leq m$

$$\begin{cases} p_{-1}^i = 1, p_0^i = 0, p_1^i = \lambda_i \\ q_{-1}^i = 0, q_0^i = 1, q_1^i = 1 \end{cases}$$

and for $k \geq 1$,

$$\begin{cases} p_{2k}^i = -(4k-1)p_{2k-1}^i + (2k-1)(2k)\lambda_i^2 p_{2k-2}^i \\ q_{2k}^i = -(4k-1)q_{2k-1}^i + (2k-1)(2k)\lambda_i^2 q_{2k-2}^i \end{cases}$$

$$\begin{cases} p_{2k+1}^i = -(4k-1)p_{2k}^i + (2k-1)(2k)\lambda_i^2 p_{2k-1}^i \\ q_{2k+1}^i = -(4k-1)q_{2k}^i + (2k-1)(2k)\lambda_i^2 q_{2k-1}^i \end{cases}$$

By lemma 3.1, the convergent p_n^i/q_n^i converges to $(1-\lambda_i^2)^{-1/2} arcsin\lambda_i$. It follows that P_n/Q_n converges to $(I - D2)-1/2$arcsin$D$, so that

$$\frac{\text{arcsin}\,(D)}{(I-D^2)^{1/2}} = \left[0; \frac{D}{I}, \frac{2D^2}{-3I}, \frac{2D^2}{-5I}, \frac{-(2k-2)(2k)D^2}{-(4k-1)I}, \frac{-(2k-2)(2k)D^2}{-(4k+1)I}\right]_{k=2}^{+\infty}$$

Then, we multiply the continued fraction $\frac{\text{arcsin}\,(D)}{(I-D^2)^{1/2}}$ by $(I-D^2)^{1/2}$ in the left to obtain

$$\text{arcsin }(D) = \left[0; \frac{D}{(I-D)^{-1/2}}, \frac{2D^2(I-D^2)^{1/2}}{-3I}, \frac{2D^2}{-5I}, \frac{(2k-1)(2k)D^2}{-(4k-1)I}, \frac{-(2k-1)(2k)D^2}{-(4k+1)I}\right]_{k=2}^{+\infty}$$

by proposition 2.3, we get

$$\text{X}(\text{arcsin}(D))X^{-1} = \left[0; \frac{DX^{-1}}{(I-D)^{-1/2}X^{-1}}, \frac{2D^2(I-D^2)^{1/2}X^{-1}}{-3I}, \frac{2D^2}{-5I}, \frac{(2k-1)(2k)D^2}{-(4k-1)I}, \frac{-(2k-1)(2k)D^2}{-(4k+1)I}\right]_{k=2}^{+\infty}$$

Let us define the sequence $(X_n)_{n\geq -1}$ by

$$\begin{cases} X_{-1} = X_0 = I \\ X_n = X(I-D^2)^{1/2},\ for\ n \geq 1. \end{cases}$$

Then

$$\begin{cases} \dfrac{X_1(DX^{-1})X_{-1}^{-1}}{X_1((I-D^2)^{\frac{1}{2}}X^{-1})X_0^{-1}} = \dfrac{A(I-A^2)^{1/2}}{I} \\ \dfrac{X_2(2D^2(I-D)^{-\frac{1}{2}}X^{-1})X_0^{-1}}{X_2(-3I)X_1^{-1}} = \dfrac{2A^2}{-3I} \end{cases}$$

For $k \geq 3$, we have

$$\frac{X_k(2k-1)(2k)D^2X_{k-2}^{-1}}{-X_k(4k-1)X_{k-1}^{-1}} = \frac{(2k-1)(2k)A^2}{-(4k-1)I}.$$

By applying the result of proposition 2.5 to the sequence $(X_n)_{n\geq -1}$, we finish the proof of theorem 3.2.

Before giving continued fraction expansions of arccos(A) and arcch(A), we begin with the real case in the following lemma

Lemma 3.4. *(see [7]) Let x be a real number such that 0 < x < 1, Then there holds*

$$\frac{\text{arccos }(x)}{(I-x^2)^{1/2}} = \left[0; \frac{x}{I}, \frac{(2k-1)(2k)x^2}{-(4k-1)}, \frac{(2k-1)(2k)x^2}{-(4k+1)}\right]_{k=1}^{+\infty} \qquad (3.4)$$

Now we establish a main theorem which, is a matrix version of the lemma 4.

Theorem 3.5. *Let $A \in M_m$ be a positive definite matrix such that $\|A\| < 1$. Then a continued fraction expansion of arcsin(A) is*

$$\text{arccos }(A) = \left[0; \frac{A(I-A^2)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{-(4k-1)I}, \frac{(2k-1)(2k)A^2}{-(4k+1)I}\right]_{k=1}^{+\infty} \qquad (3.5)$$

since $arcch(A) = iarccos(A)$, by vertu of proposition 2.6 and theorem 3.5, we have:

Corollary 3.6. *A continued fraction expansion of arcch(A) is given by*

$$\text{arcch }(A) = \left[0; \frac{A(A^2-I)^{1/2}}{I}, \frac{(2k-1)(2k)A^2}{(4k-1)I}, \frac{(2k-1)(2k)A^2}{(4k+1)I}\right]_{k=1}^{+\infty} \qquad (3.6)$$

With a similar method as in theorem 3.2, we prove the result of this theorem.

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A Remark on A Fundamental System of Units of Numbers Fields of degree 2, 3, 4 and 6

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Abstract: Let $M_n = (D_n)^n \pm D_n > 1$ where $D_n = tv^n \pm 1 \neq 0, t, v \in N^*$ and $n \in \{4,6\}$. The integer M_n is always written as $M_n = v^n m_n$, where m_n is a non-zero positive integer; assuming m_n square-free, we exhibit a fundamental system of units for families of pure fields $K_n = \mathbb{Q}(\sqrt[n]{M_n})$, including a family already given by H.-J. Stender.

Keywords: Fundamental system of units (FSU), Parametrization, the integral basis.

1. Introduction

There is a closed link between a fundamental system of units of some number fields, the resolution of some Diophantine equations, the cycle of continued fractions, and certain protocols in cryptography, see J. Buchmann [2]. Also, the regulator of a number field K, based on knowledge of a system fundamental of units, is essential to compute the class number of K, and therefore the Hilbert class towers and the construction of a codes on this number field (see V. Guruswami [5]. This, in addition to many other applications, justifies the study of such a system.

If K is an algebraic extension of degree $n = r + 2s$ on $\mathbb{Q}$, the field of rational numbers, where r is the number of real embeddings and $2s$ is the number of complex embeddings of K, Dirichlet (1840) established that the unit group U_k of K is generated by $r + s - 1$ units. The group U_K is said to be of rank $r + s - 1$. The set $S = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{r+s-1}\}$ of all generators, form what is called a fundamental system of units of the field K. However, the explicit determination of such a system is very limited.

The methods for determining a fundamental system of units of a number field K are very varied. However, regardless to the method adopted, the way followed by several mathematicians is to find in the field K

(1) Units,

(2) an independent system of units

(3) a maximal independent system of units,

(4) a fundamental system of units.

Such a program can be illustrated as follows: L. Bernstein and H. Hasse [1] considered the field $K = \mathbb{Q}(\omega)$, where $\omega = \sqrt[n]{D^n \pm d}$, with $d|D$ and they gave a system of units. The result was generalized by F. Halter-Koch and H.-J. Stender [6] for $d|D^n$. Based on a work of G. Frei and C. Levesque [4] that ensures the maximality of this system for $n \in \{2,3,4,6\}$, H-J Stender studied:

(1) In [11] (page 211), the case $n = 4$, where he assumes that $D^4 \pm d$ is squarefree.
(2) In [13], the case $n = 4$ where he assumes that $D^4 \pm d$ is free of power fourth.
(3) In [12] (page 87), the case $n = 6$, where he assumes that $D^6 \pm d$ is squarefree.

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These assumptions allow him to use directly the Bernstein and H. Hasse units [1] to determine a fundamental unit of the quadratic fields $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$ and $K_{2,6} = \mathbb{Q}(\sqrt{M_6})$ and a fundamental unit of the pur cubic field $K_3 = \mathbb{Q}(\sqrt[3]{M_6})$ hence the author determines then a fundamental system of units of the fields $K_6 = \mathbb{Q}(\sqrt[6]{M_6})$ and $K_4 = \mathbb{Q}(\sqrt[4]{M_4})$.

Question: What happens if M_n contains one nth power?

To partially answer to this question, (based on an idea of *C. Levesque, laval University, Quebec- Canada)*, we introduce the parameterizations:

$$M_n = (D_n)^n \pm D_n > 1 \ \ with \ \ D_n = tv^n \pm 1 \neq 0; t, v \in \mathbb{N}^*$$

Here the plus sign commutes with the minus sign in the expression of M_n and D_n , that is to say:

$$\begin{cases} Case \ \ "-" \ \ : M_n = (D_n)^n - D_n \ and \ \ D_n = tv^n + 1, \\ Case \ \ "+" \ \ : M_n = (D_n)^n + D_n \ and \ \ D_n = tv^n - 1 \end{cases}$$

Let

$$m_6 = ab > 1$$

where

$$(a, b) = \begin{cases} (tv^6 + 1, t^5v^{24} + 5t^4v^{18} + 10t^3v^{12} + 10t^2v^6 + 5t) \ in \ Case \ " - " \\ (t^5v^{24} - 5t^4v^{18} + 10\ t^3v^{12} - 10t^2v^6 + 5t, tv^6 - 1) \ in \ Case \ " + " \end{cases}$$

And let

$$m_4 = cd > 1$$

where

$$(c, d) = \begin{cases} (tv^4 + 1, t^3v^8 + 3t^2v^4 + 3t) \ in \ Case \ " - " \\ (t^3v^8 - 3t^2v^4 + 3t, tv^4 - 1) \ in \ Case \ " + " \end{cases}$$

In both cases, we have the form $M_n = m_n v^n, (n \in \{4,6\})$. In the following, we assume that m_n is square-free, but the M_n always, contains an nth power, $(n \in \{4,6\})$, unless $v = 1$, (the case $v = 1$ coincides with the case of Stender. In the following we always assume $v \geq 2$). Obviously, $K_n = \mathbb{Q}(\sqrt[n]{M_n}) = \mathbb{Q}(\sqrt[n]{m_n})$ but m_n no longer admits a parametrization similar to that of M_n, therefore the Bernstein units [1] are no longer valid. In this paper, we determine a fundamental systems of units of the number fields

$$K_n = \mathbb{Q}(\sqrt[n]{M_n}), \ \ n \in \{4,6\} \ \ and \ \ K_3 = \mathbb{Q}(\sqrt[3]{M_6})$$

and obviously those of quadratic sub-fields $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$ and $K_{2,6} = \mathbb{Q}(\sqrt{M_6})$.

In T. Nagell [7], T. Nagell [8] and H.-J. Stender [15] we find a full theory dealing with the Diophantine equations of the form $S_C: AX^2 - BY^2 = C, (C \in \{1,2,4\})$, in connection with the fundamental unit of a quadratic field; for C = 1, we summarize (see [15], theorem 3, page 295):

Theorem 1.1 *Given a solution (x, y) of the Diophantine equation $S_1: AX^2 - BY^2 = 1, A, B \in \mathbb{N}, (A, B) = 1$ and AB is square-free, such that*

$$x < \frac{1}{4}(A + B) - \frac{1}{2} \ or \ y < \frac{1}{4}(A + B) + \frac{1}{2},$$

then

$$\eta = (x\sqrt{A} + y\sqrt{B})^2$$

is a fundamental unit>1 of positive norm of the field $K = \mathbb{Q}(\sqrt{AB})$.

Now we give the main results of this section.

Theorem 1.2 *Let t; v be two nonzero positive integers,* $D_6 = tv^6 \pm 1 \neq 0$. *Let*

$$M_6 = (D_6)^6 \pm D_6 = m_6 v^6 > 1, \qquad \omega = \sqrt[6]{m_6}$$

Suppose that m_6 *is square-free. Then*

$$\eta_{2,6} = \frac{D_6}{(v^3 w^3 - (D_6)^3)^2}$$

is a fundamental unit of

$$K_{2,6} = \mathbb{Q}(\sqrt{M_6})$$

Proof: Consider the equation

$$S_1 : aX^2 - bY^2 = 1$$

First of all $(a, b) = 1$, indeed:

Case "-": Let d an integer such that $d|\alpha$ and $d|b = t[(tv^6+1)^4 + (tv^6+1)^3 + (tv^6+1)^2 + (tv^6+1) + 1] = t(a^4 + a^3 + a^2 + a + 1)$. Then $d|(b - t(a^4 + a^3 + a^2 + a) = t$.

an then $d|(a - tv^6) = 1$. Thus $(a, b) = 1$.

Case "+": Let d an integer such that $d|b$ and $d|a = t[(tv^6-1)^4 - (tv^6-1)^3 + (tv^6-1)^2 - (tv^6-1) + 1] = t(b^4 - b^3 + b^2 - b + 1)$. Then $d|(a - t(b^4 - b^3 - a^2 + b) = t$.

$d|(b - tv^6) = 1$. Thus $(a, b) = 1$. In addition the equation (S_1) has the solution,

$$(x, y) = \begin{cases} ((tv^6+1)^2, v^3) \ in \ (Case" -"), with: \\ \frac{1}{4}(a+b) - \frac{1}{2} > \frac{1}{4}(10t^3v^{12} + 10t^2v^6 + 5t - 1) > (tv^6+1)^2 = x \\ or \\ (v^3, (tv^6-1)^2) \ in \ (Case \ "+"), \text{with:} \\ \frac{1}{4}(a+b) - \frac{1}{2} = \frac{1}{4}(t^5v^{24} - 5t^4v^{18} + 10t^3v^{12} - 10t^2v^6 + tv^6 + 5t - 1) - \frac{1}{2} > v^3 = x. \end{cases}$$

So in both cases, and by theorem 1.1,

$$\eta_{2,6} = \frac{D_6}{(v^3 w^3 - (D_6)^3)^2}$$

is the fundamental unit of the quadratic field $K_{2,6} = \mathbb{Q}(\sqrt{M_6})$.

Theorem 1.3 *Let t; v be two nonzero positive integers,* $D_4 = tv^4 \pm 1 \neq 0$. *Let*

$$M_4 = (D_4)^4 \mp D_4 = m_4 v^4 > 1, \qquad \omega = \sqrt{m_4}$$

Suppose that m_4 *is square-free.Then*

$$\eta_{2,4} = \frac{(v^2 w^2 - (D_4)^2)^2}{D_4}$$

is a fundamental unit of

$K_{2,4} = \mathbb{Q}(\sqrt{M_4})$.

Proof: Consider the equation

$$S_1: cX^2 - dY^2 = 1$$

First of all $(c, d) = 1$, indeed:

Case "-": Let l an integer such that $l|c$ and $l|d = ta^2 + ta + t$; then $l|(b - ta^2 - ta) = t.$

But $l|a$, Then $l|(a - tv^2) = 1$.

Case "+": is such. In addition the equation (S_1) has the solution,

$$(x, y) = \begin{cases} (tv^4 + 1, v^2)\ in\ (Case\ \text{"-"}), with: \\ 2(a + b) - 1 = 2t^3v^8 + (6t^2 + 2t)v^4 + 6t + 1 > tv^6 + 1 = x_1 \\ or \\ (v^2, tv^4 - 1)\ in\ (Case\ \text{"+"}), \text{ with:} \\ 2(a + b) - 1 = 2t^3v^8 + (2t - 6t^2)v^4 - 6t - 3 > v^2 = x_2. \end{cases}$$

So in both cases, and by theorem 1.1,

$$\eta_{2,4} = \frac{(v^2w^2 - (D_4)^2)^2}{D_4}$$

is the fundamental unit of $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$.

2. A Fundamental System of Units of $K_3 = \mathbb{Q}(\sqrt[3]{M})$

Let the Diophantine equation

$$(G) = Ax^3 - By^3 = 1$$

with $A, B \epsilon \mathbb{N}$, square-free, $AB > 1$. According to Stender [14], we have two possibilities for the fundamental unit of $\mathbb{Q}(\sqrt[3]{AB^2})$:

Theorem 2.4 *Let $A > 1$ and $B > 1$. Let (x, y) be a solution of the equation (G). Then*

$$\eta = (x\sqrt[3]{A} - y\sqrt[3]{B})^3$$

is either a fundamental unit, or the square of the fundamental unit of the field $K = \mathbb{Q}\left(\sqrt[3]{AB^2}\right)$.

Now we give the main results of this section.

Theorem 2.5 *Let t, v be two nonzero positive integers $D_6 = tv^6 \pm 1 \neq 0$. Let*

$$M_6 = (D_6)^6 \mp D_6 = m_6v^6 > 1, and \quad \omega = \sqrt[6]{m_6}.$$

Suppose that m_6 is square-free. Then

$$\eta_3 = \pm\frac{((D_6)^2 - v^2w^2)^3}{D_6}$$

is either a fundamental unit, or the square of the fundamental unit of the field $K_3 = \mathbb{Q}\left(\sqrt[3]{M_6}\right)$.

Proof: Case "-": Let the equation

$$(G): a^2x^3 - by^3 = 1,$$

which has the solution

$$(x, y) = (tv^6 + 1, v^2),$$

Case "+": Let the equation

$$(G): ax^3 - b^2y^3 = 1,$$

which has the solution

$$(x, y) = (v^2, tv^6 - 1).$$

In both cases and bay theorem 2.4,

$$\eta_3 = \mp \frac{(v^2w^2 - (D_6)^2)^3}{D_6}$$

is the fundamental unit, or the square of the fundamental unit of the field K_3.

Let M be a positive integer cube free, then we set $M = fg^2$, with $(f, g) = 1$, $\bar{M} = f^2g, \Omega = \sqrt[3]{M}, et\ \bar{\Omega} = \sqrt[3]{\bar{M}}$

We say that

(1) $K = \mathbb{Q}(\sqrt[3]{M})$ is of first kind if

$$fg^2 \not\equiv \pm 1 (mod\ 9)$$

(2) $K = \mathbb{Q}(\sqrt[3]{M})$ is of second kind if

$$fg^2 \equiv \pm 1 (mod\ 9)$$

and by Dedekind [3], we have

Proposition 2.6 *(i) If K is of first kind, then* $\{1, \Omega, \bar{\Omega}\}$ is an integral basis of $K = \mathbb{Q}(\Omega)$.

(ii) If K is of second kind, then $\{\frac{1}{3}(1 + f\Omega + g\bar{\Omega}), \Omega, \bar{\Omega}\}$ is an integral basis of $K = \mathbb{Q}(\Omega)$. Moreover each algebraic integer of $K = \mathbb{Q}(\Omega)$ can be written in the form $\frac{1}{3}(x + y\Omega + z\bar{\Omega})$, $x, y, z \in \mathbb{Z}$.

Now, and more precisely, the fundamental unit of the field $K_3 = \mathbb{Q}(\sqrt[3]{M_6})$ is given by

Theorem 2.7 *Let t; v be two nonzero positive integers, $D_6 = tv^6 \pm 1 \neq 0$. Let*

$$M_6 = (D_6)^6 \pm D_6 = m_6v^6 > 1, and \quad \omega = \sqrt[6]{m_6}.$$

Suppose that m_6 is square-free. Then

$$\eta_3 = \pm \frac{((D_6)^2 - v^2w^2)^3}{D_6}$$

is a fundamental unit of the field $K_3 = \mathbb{Q}(\sqrt[3]{M_6}) = \mathbb{Q}(\omega^2)$.

Proof: As m_6 is square free, according to the proposition 2.6, $\{1, \omega^2, \omega^4\}$ is an integral basis of $K_3 = \mathbb{Q}(\omega^2)$ if K_3 is of first kind; and $\{\frac{1}{3}(1 + f\omega^2 + \omega^4), \omega^2, \omega^4\}$ is an integral basis of $K_3 = \mathbb{Q}(\omega^2)$ if K_3 is of second kind. In addition, according to the proposition 2.6, each algebraic integer of $K_3 = \mathbb{Q}(\omega^2)$ can be written in the form

$$\frac{1}{3}(x + y\omega^2 + z\omega^4), \quad with \quad x, y, z \in \mathbb{Z}$$

(1) Case"-": $m_6 = D_6(t^5v^{24} + 5t^4v^{18} + 10t^3v^{12} + 10t^2v^6 + 5t)\ et$

$$\eta_3 = 1 - (3(D_6)^3v^2)\omega^2 + (3D_6v^4)\omega^4 \tag{2.1}$$

Suppose that $\eta_3 = \zeta^2$, where ζ is a unit of K_3.

(a) Let K_3 is of first kind. Then

$$\zeta = x + y\omega^2 + z\omega^4, \qquad with\ x, y, z \in \mathbb{Z}$$

as $\eta_3 = \zeta^2$, we have

$$x^2 + 2yzm_6 = 1 \tag{2.2}$$
$$2xy + z^2m_6 = -3(D_6)^3v^2 \tag{2.3}$$
$$2xz + y^2 = 3D_6v^4 \tag{2.4}$$

Let's show that

$$(*) \qquad xy > 0 \quad and \quad y \neq 0.$$

According to (2.3), $x \neq 0$ and $y \neq 0$ In addition $z \neq 0$; Indeed, suppose $z = 0$; then according to (2.2), $x = \pm 1$; according to (2.3), $2y = \pm 3(D_6)^3v^2$, i.e. $4y^2 = 9(D_6)^6v^4$; but according to (2.4), $4y^2 = 12D_6v^4$; then we have $12D_6v^4 = 9(D_6)^6v^4$, i.e. 3|4, a contradiction. According to (2.3), $xy < 0$, and according to (2.2), $yz < 0$. Then x and z have the same sign, i.e. $xz > 0$.

According to (2.2), we have

$$(**) \qquad (x, m_6) = 1.$$

Then $(x, D_6) = 1$. According to (2.3), $D_6|2xy$, i.e. $D_6|2y$. Then (2.4) becomes

$$8xz + (2y)^2 = 12D_6v^4 \tag{2.5}$$

Then $(D_6)^2|(8z^2)$. And (2.3) becomes

$$2(8)^2xy + (8z)^2m_6 = -3(8)^2(D_6)^3v^2 \tag{2.6}$$

Then $(D_6)^3|2(8)^2y$, since $D_6|m_6$. And (2.4) becomes

$$(2^2)(8^4)2xz + (2(8^2)y)^2 = (2^2)(8^4)3D_6v^4 \tag{2.7}$$

But $D_6|8z$; then, (seeing that $xz > 0$),

$$(2^2)(8^4)2xz = (8^4)xz_1D_6 > 0$$

But $(D_6)^6|(2(8^2)y)^2$; then

$$(2(8^2)y)^2 = (y_1)^2(D_6)^6 > 0$$

But

$$(2^2)(8^4)3D_6v^4 < (2^2)(8^4)3(D_6)^2$$

Then (2.7) implies

$$(8^4)xz_1D_6 + (y_1)^2(D_6)^6 < (2^2)(8^4)3D_6)^2 \tag{2.8}$$

which is impossible for $D_6 \geq 16$; but $v \geq 2$, whereby that $D_6 = tv^6 + 1 > 2^6 = 64$.

(b) Let K_3 of the second kind. Then

$$\zeta = \frac{1}{3}(x + y\omega^2 + z\omega^4), \quad with \quad x, y, z \in \mathbb{Z}$$

As $\eta_3 = \zeta^2$, we have

$$x^2 + 2yzm_6 = 9 \tag{2.9}$$

$$2xy + z^2 m_6 = -27(D_6)^3 v^2, \tag{2.10}$$

$$2xz + y^2 = 27D_6 v^4, \tag{2.11}$$

Then $(x, m_6) = 1, 3\ ou\ 9$. The 9 is excluded because m_6 is square free. whether $(x, m_6) = 1$ We have then the propriete (**) of first case, and get the equivalent of (2.8), namely

$$(8^4)xz_1 D_6 + (y_1)^2 (D_6)^6 < (2^2)(8^4)27(D_6)^2$$

which is impossible for $D_6 \geq 27$, i.e. for all $v \geq 2$.

Whether $(x, m_6) = 3$ according to (2.9), $3|y$ or $3|z$. If $3|y$, then according to (2.10), $3|z$. If $3|z$, then according to (2.11), $3|y$. Brief, $3|y$ and $3|z$. Let

$$x_1 = \left(\frac{x}{3}\right), \quad y_1 = \left(\frac{y}{3}\right) \quad et \quad z_1 = \left(\frac{z}{3}\right)$$

Then

$$x_1^2 + 2y_1 z_1 m_6 = 1. \tag{2.12}$$

$$2x_1 y_1 + z_1^2 m_6 = -3(D_6)^3 v^2 \tag{2.13}$$

$$2x_1 z_1 + y_1^2 = 3(D_6) v^4 \tag{2.14}$$

Which brings us back again to the same contradiction above.

(2) Case "+": As

$$v^6 m_6 = (D_6)^6 + D_6 \tag{2.15}$$

We derive

$$m_6 > \frac{(D_6)^6}{v^6} \tag{2.16}$$

Furthermore,

$$\eta_3 = 1 + (3(D_6)^3 v^2)\omega^2 - (3D_6 v^4)\omega^4 \tag{2.17}$$

Suppose that $\eta_3 = \zeta^2$, We distinguish two cases.

(a) Let K_3 be of the first kind. Then

$$\zeta = x + y\omega^2 + z\omega^4, \quad with \;\; x, y, z \in \mathbb{Z}$$

Then

$$x^2 + 2yzm_6 = 1 \tag{2.18}$$

$$2xy + z^2m_6 = 3(D_6)^3v^2 \tag{2.19}$$

$$2xz + y^2 = -3D_6v^4 \tag{2.20}$$

Let's show that

$$(*') \qquad (x, m_6) = 1 \quad et\ D_6|4z$$

According to (2.18), $(x, m_6) = 1$. According to (2.19), $D_6|2y$. Then (2.20) becomes

$$4xz + 2y^2 = -6D_6v^4 \tag{2.21}$$

and $D_6|4z$

Let's show that

$$(**') \qquad xy > 0 \quad et \;\; z \neq 0$$

We have $x \neq 0$ and $z \neq 0$ otherwise according to (2.20), $y^2 = -3v^4D_6 < 0$. In addition $y \neq 0$; suppose the contrary; according to (2.19), $(4z)^2m_6 = (4^2)(3)v^2(D_6)^3$; according to $(*')$, $4z = z_1D_6$; according to (2.16), $m_6 > \frac{(D_6)^6}{v^6}$, Then

$$(4^2)(3)v^2(D_6)^3 = (4z)^2m_6 > z_1^2\left(\frac{(D_6)^8}{v^6}\right)$$

which is impossible for $v \geq 2$. According to (2.18), $x^2 = 1 - 2yzm_6 > 0$ then $yz < 0$. According to (2.20), $2xz = -3v^4D_6 - y^2 < 0$ then $xz < 0$. Then $xy > 0$ The equation (2.19) becomes

$$(4^2)2xy + (4z)^2m_6 = (4^2)3(D_6)^3v^2 \tag{2.22}$$

Then

$$(4^2)3(D_6)^3v^2 = (4^2)2xy + z_1^2D^2m_6 > z_1^2\left(\frac{(D_6)^8}{v^6}\right)$$

which is impossible for $v \geq 2$.

(b) Let K_3 be of the second kind. Then

$$\zeta = \frac{1}{3}(x + y\omega^2 + z\omega^4), \quad with \;\; x, y, z \in \mathbb{Z}$$

As $\eta_3 = \zeta^2$, we have

$$x^2 + 2yzm_6 = 9 \tag{2.23}$$

$$2xy + z^2m_6 = 27(D_6)^3v^2 \tag{2.24}$$

$$2xy + y^2 = -27D_6v^4 \tag{2.25}$$

According to (2.23), $(x, m_6) = 1,3$ ou 9. 9 is excluded. Whether $(x, m_6) = 1$. We have then the property $(*')$ and we deduce a contradiction as above. Let $(x, m_6) = 3$. According to (2.23), $3|yz$, If $3|y$, then according to (2.24), $3|z$. If $3|z$, then according to (2.25), $3|y$. Let

$$x_1 = \left(\frac{x}{3}\right), \quad y_1 = \left(\frac{y}{3}\right) \quad et\ z_1 = \left(\frac{z}{3}\right);$$

and we deduce a contradiction as above.

3. A Fundamental System of Units of $K = \mathbb{Q}(\sqrt[6]{M_6})$

We have m_6 is square-free, the field $K_6 = \mathbb{Q}(\omega), \omega = \sqrt[6]{m_6}$, is of degree 6 over $\mathbb{Q}$, in addition it admits a quadratic sub-field $K_{2,6} = \mathbb{Q}(\omega^3)$ with fundamental unit $\eta_{2,6}$ (theorem 1.2), and a cubic sub-field $K_3 = \mathbb{Q}(\omega^2)$ with fundamental unit η_3 (theorem 2.7). For the determination of a fundamental system of units of the field $K_6 = \mathbb{Q}(\sqrt[6]{M_6})$, we use the Stender theorem [12]:

Theorem 3.8 *Let $\eta_{2,6}$ be the fundamental unit of $K_{2,6}$, and let η_3 be the fundamental unit of K_3. Let $\xi_2, \xi_3 \epsilon K_6$ such that*

$$N_{K_6/K_{2,6}}(\xi_2) = \eta_{2,6}, \quad N_{K_6/K_3}(\xi_2) = \pm 1$$

and

$$N_{K_6/K_{2,6}}(\xi_3) = 1, \quad N_{K_6/K_3}(\xi_2) = \pm \eta_{3,6}$$

Let $\epsilon_1 \in K_6$ be the smallest unit>1, satisfying:

$$N_{K_6/K_{2,6}}(\epsilon_1) = 1, \quad N_{K_6/K_3}(\epsilon_1) = \pm 1$$

Then

$$\{\xi_2, \xi_3, \epsilon_1\}$$

is a fundamental system of d'units of K_6.

Let ϱ be a third root of unity, $(\varrho^2 + \varrho + 1 = 0)$; the conjugates $\alpha^{(j)}$ of an algebraic integer α of field $K_6, 0 \leq j \leq 5$, are given by:

$$\begin{cases} \alpha^{(0)} = \alpha \\ \alpha^{(1)} = -\alpha \\ \alpha^{(2)} = \varrho\alpha \\ \alpha^{(3)} = \varrho^2\alpha \\ \alpha^{(4)} = -\varrho\alpha \\ \alpha^{(5)} = -\varrho^2\alpha \end{cases} \tag{3.26}$$

And according to Stender [12], the product $\alpha\omega$ can be written in the form:

$$\alpha\omega = \frac{1}{6}(x_0 + x_1\omega + x_2\frac{\omega^2}{h} + x_3\frac{\omega^3}{h}x_4\frac{\omega^4}{gh^2} + x_5\frac{\omega^5}{gh^2}) \tag{3.27}$$

with $x_i \epsilon \mathbb{Z}, 0 \leq i \leq 5$.

Remark 3.9 *Since m_6 is square free, then we can take $g = h = 1$ in (3.27), (see [12] page 80 and page 87).*

In addition, we have:

Proposition 3.10 *Let α be an algebraic integer of the field $K_6 = \mathbb{Q}(\omega)$. Let β be a unit >1 such that*

$$N_{K_6/K_{2,6}}(\beta) = 1, \quad N_{K_6/K_3}(\beta) = \pm 1$$

Suppose that $\beta = \alpha^n$. Then

$$|x_i| < \frac{k_i}{\omega^{i-1}}\left(\sqrt[n]{\beta} + 2\sqrt[n]{|\beta^{(4)}|} + 3\right), \qquad 0 \le i \le 5; \tag{3.28}$$

where $k_0 = k_1 = 1, k_2 = k_3 = h, k_4 = k_5 = gh^2$. In addition $x_4 \neq 0$ and $x_5 \neq 0$.

Now we give the main results of this section.

Theorem 3.11 *Let t, v be two nonzero integers, $D_6 = tv^6 \mp 1 > 0$. Let*

$$M_6 = (D_6)^6 \pm D_6 = m_6 v^6 > 1, and \quad \omega = \sqrt[6]{m_6}.$$

Suppose that m_6 is square-free. Then

$$\{\xi_2 = \pm\frac{v\omega + D_6}{v\omega - D_6}, \xi_3 = \frac{v^3\omega^3 - (D_6)^3}{(v\omega - \mathrm{D}_6)^3}, \qquad \epsilon_1 = \xi_2^3 {\eta_{2,6}}^{-1}\}$$

is a fundamental system of units of $K_6 = \mathbb{Q}\left(\sqrt[6]{M_6}\right)$

Proof: ξ_2 and ξ_3 satisfies theorem 3.8, namely:

$$N_{K_6/K_{2,6}}(\xi_2) = \eta_{2,6}, \quad N_{K_6/K_3}(\xi_2) = \pm 1$$

And

$$N_{K_6/K_{2,6}}(\xi_3) = 1, \quad N_{K_6/K_3}(\xi_3) = \pm\eta_3$$

For

$$\epsilon_1 = \xi_2^3 {\eta_{2,6}}^{-1} = \xi_3^2 {\eta_3}^{-1} = \xi_6 {\eta_3}^{-1} {\eta_{2,6}}^{-1}$$

where

$$\xi_6 = \frac{D_6}{(v\omega - D_6)^6}$$

we have

$$\epsilon_1 > 1, \qquad N_{K_6/K_{2,6}}(\epsilon_1) = 1, \quad et \quad N_{K_6/K_3}(\epsilon_1) = \pm 1 \tag{3.29}$$

Let’s show that ϵ_1 is the smallest unit that verifies (3.29):

Lemma 3.12 *(i) In Case"-", we have*

$$\begin{cases} \dfrac{4v^6\omega^6}{D_6} < \eta_{2,6} < 4(D_6)^5, \\ \dfrac{12v^6\omega^6}{D_6} < \xi_2 < 12(D_6)^5 \end{cases}$$

(ii) In Case “+”, we have

$$\begin{cases} 4(D_6)^5 < \eta_{2,6} < \dfrac{4v^6\omega^6}{D_6}, \\ 12(D_6)^5 < \xi_2 < \dfrac{12v^6\omega^6}{D_6} \end{cases}$$

Proof: (i) Case “-“:

$$D_6 - 1 < v\omega < D_6.$$

Since

$$D_6 = (D_6)^6 - (v\omega)^6,$$

we deduce

$$\frac{D_6}{(D_6)^3 - (v\omega)^3} = (D_6)^3 + (v\omega)^3$$

and

$$\frac{D_6}{D_6 - v\omega} = ((D_6)^5 + v\omega(D_6)^4 + v^2\omega^2(D_6)^3 + v^3\omega^3(D_6)^2 + v^4\omega^4 D_6 + v^5\omega^5).$$

We have

$$\eta_{2,6} = \frac{D_6}{(v^3\omega^3 - (D_6)^3)^2} = \frac{1}{D_6}(\frac{D_6}{v^3\omega^3 - (D_6)^3})^2 = \frac{1}{D_6}((D_6)^3 + v^3\omega^3)^2$$

then

$$\frac{4v^6\omega^6}{D_6} < \eta_2 < 4(D_6)^5.$$

Similarly

$$\xi_2 = -\frac{v\omega + D_6}{v\omega - D_6} = \frac{v\omega + D_6}{D_6}((D_6)^5 + v\omega(D_6)^4 + v^2\omega^2(D_6)^3 + v^3\omega^3(D_6)^2 + v^4\omega^4 D_6 + v^5\omega^5)$$

Then

$$\frac{12v^6\omega^6}{D_6} < \xi_2 < 12(D_6)^5.$$

(ii) Case “+”:$v\omega > D_6$, and just swap D_6 and $v\omega$.

Lemma 3.13

$$1 < \epsilon_1 < \begin{cases} 3^3 2^4 \dfrac{(D_6)^{16}}{v^6\omega^6} \;\; Case\; "-", \\ 3^3 2^4 \dfrac{v^{18}\omega^{18}}{(D_6)^8} \;\; Case\; "+", \end{cases}$$

Proof: Case “-“

$$\epsilon_1 = \xi_2^3 {\eta_2}^{-1} < (12(D_6)^5)^3 \left(\frac{D_6}{4(v\omega)^6}\right) = 3^3 2^6 \left(\frac{(D_6)^{16}}{4(v\omega)^6}\right) = 3^3 2^4 \left(\frac{(D_6)^{16}}{(v\omega)^6}\right)$$

On the other hand, according to lemma 3.12,

$$\epsilon_1 = \xi_2^3 \eta_2^{-1} > \left(\frac{12v^6\omega^6}{D_6}\right)^3 \, (4(D_6)^5)^{-1} > 1$$

In Case"+", we use lemma 3.12 and the fact that $v\omega > D_6$.

Lemma 3.14 *(i) Case "-":*

$$1 < \left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| < 12\sqrt{3}\left(\frac{(D_6)^8}{v^3\omega^3}\right)$$

(ii) Case "+":

$$1 < \left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| < 12\sqrt{3}\left(\frac{v^9\omega^9}{(D_6)^3}\right)$$

Proof: According to (3.26), $\epsilon_1^{(4)} = \overline{\epsilon_1^{(5)}}$. Then $\left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right|$. On the other hand,

$$\left|\epsilon_1^{(4)}\right| = \left|\left(\xi_2^{(4)}\right)^3\left(\eta_{2,6}^{(4)}\right)^{-1}\right| = \left|\xi_2^{(4)}\right|^3\left|\eta_{2,6}^{(4)}\right|^{-1}$$

and

$$\left|\xi_2^{(4)}\right|^2 = \xi_2^{(4)}\left(\overline{\xi_2^{(4)}}\right) = \xi_2^{(4)}\xi_2^{(5)} = \frac{(D_6)^2 + D_6 v\omega + v^2\omega^2}{(D_6)^2 - D_6 v\omega + v^2\omega^2} > 1.$$

Then

$$\left|\xi_2^{(4)}\right| > 1 \quad and \ \left|\xi_2^{(4)}\right|^3 > 1$$

We have

$$1 < \eta_{2,6} = \left|\eta_{2,6}{}^{(1)}\right|^{-1} = \left|\eta_{2,6}{}^{(4)}\right|^{-1} = \left|\eta_{2,6}{}^{(5)}\right|^{-1}$$

Then

$$\left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| = \left|\xi_2^{(4)}\right|^3\left|\eta_{2,6}{}^{(4)}\right|^{-1} > 1$$

On the other hand,

$$\left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| = \left(\xi_2^{(4)}\xi_2^{(5)}\right)^{3/2}\left(\eta_{2,6}{}^{(4)}\eta_{2,6}{}^{(5)}\right)^{-1/2}$$

$$= \left(\frac{(D_6)^2 + D_6 v\omega + v^2\omega^2}{(D_6)^2 - D_6 v\omega + v^2\omega^2}\right)^{3/2} \frac{(v^3\omega^3 + (D_6)^3)^2}{D_6}$$

Then

Case "-": We have $v\omega < D_6$; then

$$\left|\epsilon_1^{(4)}\right| < \left(\frac{3(D_6)^2}{v^2\omega^2}\right)^{3/2}\left(\frac{4(D_6)^6}{D_6}\right) = 12\sqrt{3}\left(\frac{(D_6)^8}{v^3\omega^3}\right)$$

Case "+":We have $v\omega > D_6$; and

$$\left|\epsilon_1^{(4)}\right| < \left(\frac{3v^2\omega^2}{(D_6)^2}\right)^{3/2}\left(\frac{4v^6\omega^6}{D_6}\right) = 12\sqrt{3}\left(\frac{v^9\omega^9}{(D_6)^3}\right)$$

Lemma 3.15 *ϵ_1 is the smallest unit > 1 of field $K_6 = \mathbb{Q}(\omega)$ such that*

$$N_{K_6/K_{2,6}}(\epsilon_1) = 1, \quad et \quad N_{K_6/K_3}(\epsilon_1) = \pm 1$$

Proof: Argue by contradiction and assume that

$$\epsilon_1 = \alpha^n \quad with \ n > 1$$

There $n \notin \{2,3\}$, because $\sqrt{\eta_3} \notin K_6$ and $\sqrt[3]{\eta_{2,6}} \notin K_6$. Let $n \geq 5$. In specializing $\beta = \epsilon_1$, in the proposition 3.10, then for $i \in \{4.5\}$ we have

$$|x_i| < \left(\frac{1}{\omega^{i-1}}\right)\left(\sqrt[5]{\epsilon_1} + 2\sqrt[5]{\left|\epsilon_1^{(4)}\right|} + 3\right)$$

Case “-“:

$$\epsilon_1 = 3^3 2^4 \left(\frac{(D_6)^{16}}{v^6\omega^6}\right)$$

such that

$$\left|\epsilon_1^{(4)}\right| < 12\sqrt{3}\left(\frac{(D_6)^8}{v^3\omega^3}\right)$$

we obtain

$$|x_5| < \left(\frac{(D_6)^3}{v^4\omega^4}\right)\left(\sqrt[5]{3^3 2^4\left(\frac{D_6 v^{20}}{v^6\omega^6}\right)} + 2\sqrt[5]{12\sqrt{3}\left(\frac{v^{20}}{v^3\omega^3(D_6)^7}\right)} + \left(\frac{3v^4}{(D_6)^3}\right)\right). \tag{3.30}$$

But

$$G_1 = \sqrt[5]{3^3 2^4\left(\frac{D_6 v^{20}}{v^6\omega^6}\right)} < \sqrt[5]{3^3 2^4\left(\frac{(D_6)^5}{(D_6)^6 - D_6}\right)}$$

$$= \sqrt[5]{3^3 2^4\left(\frac{1}{D_6 - (D_6)^{-4}}\right)}$$

$$G_2 = 2\sqrt[5]{12\sqrt{3}\left(\frac{v^{20}}{v^3\omega^3(D_6)^7}\right)} < 2\sqrt[5]{12\sqrt{3}\left(\frac{(D_6)^4}{v^3\omega^3(D_6)^7}\right)}$$

$$< 2\sqrt[5]{12\sqrt{3}\left(\frac{1}{(D_6)^3}\right)}$$

$$G_3 = \frac{3v^4}{(D_6)^3} < \frac{3}{(D_6)^2},$$

$$\frac{(D_6)^3}{v^4\omega^4} < \frac{(D_6)^5}{(D_6)^6 - D_6} = \frac{1}{D_6 - (D_6)^{-4}}$$

Then $|x_5| < 1$, because $D_6 > 2^6 = 64$, and $x_5 = 0$ because x_5 is an integer. In addition,

$$|x_4| = \left(\frac{(D_6)^3}{v^3\omega^3}\right)\left(\sqrt[5]{3^3 2^4\left(\frac{D_6 v^{15}}{v^6\omega^6}\right)} + 2\sqrt[5]{12\sqrt{3}\left(\frac{v^{15}}{v^3\omega^3(D_6)^7}\right)} + \left(\frac{3v^3}{(D_6)^3}\right)\right) \tag{3.31}$$

But

$$\sqrt[5]{3^3 2^4\left(\frac{D_6 v^{15}}{v^6\omega^6}\right)} < G_1$$

$$2\sqrt[5]{12\sqrt{3}\left(\frac{v^{15}}{v^3\omega^3(D_6)^7}\right)} < G_2$$

$$\left(\frac{3v^3}{(D_6)^3}\right) < G_3$$

$$\left(\frac{(D_6)^3}{v^3\omega^3}\right) < \frac{1}{1-(D_6)^{-5}}$$

Then $x_4 = 0$.

Case "+":

$$|x_5| < \sqrt[5]{3^3 2^4\left(\frac{1}{v^{22}\omega^2}\right)} + 2\sqrt[5]{12\sqrt{3}\left(\frac{1}{v^6\omega^{11}}\right)} + \left(\frac{3}{\omega^3}\right) \tag{3.32}$$

Then in a analogous manner $x_5 = 0$. In addition,

$$|x_5| < \sqrt[5]{3^3 2^4\left(\frac{1}{v^5}\right)\sqrt{\frac{(D_6)^6 + D_6}{(D_6)^8}}} + 2\sqrt[5]{12\sqrt{3}\left(\frac{1}{v^6\omega^6}\right)} + \left(\frac{3}{\omega^3}\right) \tag{3.33}$$

Then $x_4 = 0$. This completes the prof of theorem 3.11.

4. A Fundamental System of Units of $\boldsymbol{K} = \mathbb{Q}(\sqrt[4]{\boldsymbol{M_4}})$

We assume that m_4 is square-free, the field $K_4 = \mathbb{Q}(\omega), (\omega = \sqrt[4]{m_4})$, is of degree 4 over $\mathbb{Q}$, in addition it admits a sub-quadratic field $K_{2,4} = \mathbb{Q}(\omega^2)$ with fundamental unit $\eta_{2,4}$ (theorem 1.3). We introduce here the proprieties of fields of degree 4 taken follows [9] and [11].

Every algebraic integer α of K_4 can be written as form

$$\alpha = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \quad with\ x_0, \qquad x_1, x_2, x_3 \in \mathbb{Z} \tag{4.34}$$

We denote by

$$\begin{cases} \omega = \sqrt[4]{m_4} \\ \omega^{(1)} = -\omega \\ \omega^{(2)} = i\omega \\ \omega^{(3)} = -i\omega \end{cases} \tag{4.35}$$

the four conjugates ω. replacing ω respectively by $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$ in (4.42), we get

$$\alpha^{(1)} = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \tag{4.36}$$

$$\alpha^{(2)} = \frac{1}{4}(x_0 + x_1 i\omega - x_2\omega^2 - x_3 i\omega^3) \tag{4.37}$$

$$\alpha^{(3)} = \frac{1}{4}(x_0 - x_1 i\omega - x_2\omega^2 + x_3 i\omega^3) \tag{4.38}$$

If in addition β is an algebraic integer such that $\beta = \pm\alpha^n, n \geq 1$, then

$$|x_3| \leq \left(\frac{1}{\omega^3}\right)\sum_{j=0}^{3}\sqrt{|\beta^{(j)}|} \tag{4.39}$$

Denote by ε_0 the smallest unit >1 of K_4 satisfying the property

$$\varepsilon_0\varepsilon_0^{(1)} = 1; \tag{4.40}$$

then any other unit ε of K_4 which satisfies the properties (4.40), is of the form

$$\varepsilon = \varepsilon_0^n, \;\; n \geq 1 \tag{4.41}$$

writing

$$\varepsilon_0 = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \quad \text{with } x_0, x_1, x_2, x_3 \in \mathbb{Z} \tag{4.42}$$

then according to (4.40) and (4.41) we have in addition

$$0 \neq |x_3| < \frac{1}{\omega^3}(3 + \sqrt{\varepsilon}) \tag{4.43}$$

Theorem 4.16 *Let $\eta_{2,4}$ be the fundamental unit of the quadratic field $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$, and let ε_0 be the smallest unit of K_4 satisfying $\varepsilon_0\varepsilon_0^{(1)} = 1$. If $\sqrt{(\eta_{2,4})^{-1}\varepsilon_0} \notin K_4$, then*

$$\{\eta_{2,4}, \varepsilon_0\}$$

is a fundamental system of unit of K_4.

Now we give the main results of this section.

Theorem 4.17 *Let t, v be two nonzero positive integers, $D_4 = tv^4 \mp 1$. Let*

$$M_4 = D_4^4 \pm D_4, m_4 = \frac{M_4}{v^4}, \omega = \sqrt[4]{m_4}.$$

Suppose that m_4 is square-free. Then

$$\left\{\eta_{2,4} = \frac{(v^2\omega^2 + D_4^2)^2}{D_4}, \varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4}\right\}$$

is a fundamental system of units of the quartique real field $K_4 = \mathbb{Q}(\sqrt[4]{M_4})$

Proof: Remains to verify that the unit ε_1 satis_es the property (4.40), and that K_4 is of the first kind (i.e.: $\sqrt{(\eta_{2,4})^{-1}\varepsilon_1} \notin K_4$). In fact,

$$\varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4}$$

is a unit of K_4 of norm 1 because

$$\varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4} = 2(D_4)^3 \mp 1 + 2(D_4)^2 v\omega + 2D_4 v^2\omega^2 + 2v^3\omega^3$$

is an algebraic integer such that

$$N_{K/\mathbb{Q}}(\varepsilon_1) = \left(\frac{v\omega + D_4}{v\omega - D_4}\right)\left(\frac{-v\omega + D_4}{-v\omega - D_4}\right)\left(\frac{iv\omega + D_4}{iv\omega - D_4}\right)\left(\frac{-iv\omega + D_4}{-iv\omega - D_4}\right) = 1$$

Lemma 4.18 *Let* $\varepsilon_1 = \pm\left(\frac{v\omega+D_4}{v\omega-D_4}\right)$ *et* $\varepsilon_0 = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3)$ *with* $x_i \epsilon \mathbb{Z}$*. Assume that* $\varepsilon_1 = \varepsilon_0^n$ *with* $n \geq 2$*. Then*

1. *Case "+":* $|x_3| < \frac{3}{\omega^3} + \frac{(v^3)\sqrt{8}}{D_4}$
2. *Case "-":* $|x_3| < \frac{3}{\omega^3} + \frac{(v^3 D_4)\sqrt{8D_4}}{(tv^4)^3}$

Proof: Since $v^4\omega^4 = (D_4)^4 \pm D_4$, then

$$(*_0)\begin{cases} Case\ \text{"+"}: we\ have\ D_4 < v\omega < D_4 + 1, \\ Case\ \text{"-"}: we\ have\ D_4 - 1 < v\omega < D_4 \end{cases}$$

But

$$\varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4} = 2(D_4)^3 \mp 1 + 2(D_4)^2 v\omega + 2D_4 v^2\omega^2 + 2v^3\omega^3 \tag{4.44}$$

which gives us

$$(*_1)\begin{cases} Case\ \text{"+"}: 8(D_4)^3 < \varepsilon_1 < 8\dfrac{(v\omega)^4}{D_4}\ , \\ Case\ \text{"-"}: 8\dfrac{(v\omega)^4}{D_4} < \varepsilon_1 < 8(D_4)^3 \end{cases}$$

then

$$(*_2)\begin{cases} Case\ \text{"+"}:\ \dfrac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \dfrac{\sqrt{8}}{D_4}, \\ Case\ \text{"-"}:\ \dfrac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \dfrac{(D_4)\sqrt{8D_4}}{(tv^4)^3} \end{cases}$$

in fact

Case "+": According to $(*_1)$, $\sqrt{\varepsilon_1} < \left(\sqrt{8}\right)\left(\frac{(v\omega)^2}{\sqrt{D_4}}\right)$, and then one applies $(*_0)$.

Case "-": According to $(*_1)$, $\frac{\varepsilon_1}{(v\omega)^6} < \frac{8(D_4)^3}{(v\omega)^6}$, according to $(*_0)$, $\frac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \frac{(D_4)\sqrt{8D_4}}{(D_4-1)^3}$, finally $\frac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \frac{(D_4)\sqrt{8D_4}}{(tv^4)^3}$

Then we have

$$\varepsilon_1\varepsilon_1^{(1)} = \left(\frac{v\omega + D_4}{v\omega - D_4}\right)\left(\frac{-v\omega + D_4}{-v\omega - D_4}\right) = 1$$

According to (4.43)

$$0 < |x_3| < \frac{1}{\omega^3}(3 + \sqrt{\varepsilon_1})$$

Then

$$0 < |x_3| < \frac{3}{\omega^3} + \frac{v^3\sqrt{\varepsilon_1}}{(v\omega)^3} \tag{4.45}$$

replacing $\frac{\sqrt{\varepsilon_1}}{(v\omega)^3}$, by $\frac{\sqrt{8}}{D_4}$ in Case "+", and by $\frac{(D_4)\sqrt{8D_4}}{(tv^4)^3}$ in Case "-", conclude using $(*_2)$

Lemma 4.19 *Suppose that m_4 is square free. Then*

$$\varepsilon_1 = \pm\left(\frac{v\omega + D_4}{v\omega - D_4}\right)$$

is the smallest unit of K_4 which satisfies

$$\varepsilon_1\varepsilon_1^{(1)} = 1$$

Proof: recall that $v \geq 2$. We have then

$$\varepsilon_1\varepsilon_1^{(1)} = \left(\frac{v\omega + D_4}{v\omega - D_4}\right)\left(\frac{-v\omega + D_4}{-v\omega - D_4}\right) = 1$$

Argue by contradiction. Then according to (4.41) we have,

$$\varepsilon_1 = \varepsilon_0^n, \; ou \; \varepsilon_0 = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \text{ with } x_i \epsilon \mathbb{Z}.$$

According to lemma 4.18, we have

(i) Case "+",

$$|x_3| < \frac{3}{\omega^3} + \frac{v^3\sqrt{8}}{D_4} \tag{4.46}$$

Since $\omega > 3$, then $\frac{3}{\omega^3} < \frac{1}{9}$. Then

$$|x_3| < \frac{3}{\omega^3} + \frac{v^3\sqrt{8}}{D_4} = \frac{3}{\omega^3} + \frac{v^3\sqrt{8}}{tv^4 - 1} < \frac{1}{9} + \frac{v^3\sqrt{8}}{v^4 - 1}$$

But

$$\frac{1}{9} + \frac{v^3\sqrt{8}}{v^4 - 1} < 1 \Leftrightarrow 4v^4 - (9\sqrt{2})v^3 - 4 > 0$$

This is true for $v > 3$. Then for $v > 3$

$$|x_3| < \frac{1}{9} + \frac{v^3\sqrt{8}}{v^4 - 1} < 1$$

Then $x_3 = 0$, contradiction with (4.43). the result remains true for $v \in \{2,3\}$, because for $v = 3$, just directly replace in (4.46). The same for $v = 2$ and $t \geq 2$, just replace directly in (4.46). For $(v, t) = (2,1)$, just replace directly in (4.45).

In the following, we do not treat the first two values of $v,(v=2,3)$, because the result is the same by the same argument.

(ii) Case "-",

$$|x_3|<\frac{3}{\omega^3}+\frac{v^3D_4\sqrt{8D_4}}{(tv^4)^3}<\frac{1}{9}+\left(\frac{v^3(2tv^4)(\sqrt{2}\sqrt{t}\sqrt{8})v^2}{t^3v^{12}}\right)<\frac{1}{9}+\frac{\sqrt{8}}{v^3}<1$$

Then we have the same contradiction.

We show that $\xi=\sqrt{(\eta_{2,4})^{-1}\varepsilon_1}\notin K_4$. But

$$\xi=\sqrt{(\eta_{2,4})^{-1}\varepsilon_1}=\sqrt{\frac{\varepsilon_1D_4}{(v^2\omega^2+(D_4)^2)^2}}=\frac{1}{(v^2\omega^2+(D_4)^2)^2}\sqrt{\varepsilon_1D_4}$$

If $\xi\in K_4$ then $\sqrt{\varepsilon_1D_4}\in K_4$. According to (4.42), we have

$$\sqrt{\varepsilon_1D_4}=\frac{1}{4}(x_0+x_1\omega+x_2\omega^2+x_3\omega^3) \text{ with } x_0,x_1,x_2,x_3\epsilon\mathbb{Z}.$$

Using (4.39), we have

$$|x_3|\leq\left(\frac{1}{\omega^3}\right)\sum_{j=0}^{3}\sqrt{|(D_4\varepsilon_1)^{(j)}|}$$

Then

$$|x_3|<\frac{1}{\omega^3}\left(\sqrt{D_4\varepsilon_1}+1+2\sqrt{D_4}\right)$$

According to $(*_0)$ we have

Case “+”,

$$|x_3|<\frac{1}{\omega^3}\left(\sqrt{D_4\varepsilon_1}+1+2\sqrt{D_4}\right)<\left(\frac{1}{v-\frac{1}{v^3}}+\frac{1}{(v-\frac{1}{v^3})^3}+\frac{1}{(v^4-\frac{1}{v^2})^2}\right)<1$$

Case “-“,

$$|x_3|<\frac{1}{\omega^3}\left(\sqrt{D_4\varepsilon_1}+1+2\sqrt{D_4}\right)$$

$$<\frac{v^3(D_4)^2\sqrt{8}}{(D_4)^3-3(D_4)^2+3D_4-1}+\frac{1}{\omega^3}+\frac{2v^3\sqrt{D_4}}{(D_4)^3-3(D_4)^2+3D_4-1}$$

$$<\frac{\sqrt{8}}{v-\frac{3}{v^3}+\frac{3}{D_4v^3}-\frac{1}{(D_4)^2v^3}}+\frac{1}{\omega^3}+\frac{2}{v+\frac{3}{v^3}-\frac{1}{D_4v^3}}<1$$

Then $x_3=0$; then the same contradiction arises. This completes the demonstration of theorem 4.17.

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Path Partition in Directed Graph-Modeling and Optimization

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Abstract: The concept of graph theory is therefore perfectly suitable to structure a problem in its initial analysis phases since a graph is the most general mathematical object. At the structural level, the nodes represent the objects, the variables… and the arc forms the binary relation of influence among them. Many real problems can be modeled as path partition in directed graph that played particular role in the operation of arranging a set of nodes especially in case of directed acyclic graph (DAG). We encounter such graph in schedule problems, the analysis of language structure, the probability theory, the game theory, compilers…. Moreover managerial problem can be modeled as acyclic graphs, also the potential problem has a suitable solution if and only if the graph G is acyclic.

The arc – disjoint paths in a graph has an important application in several areas and needs exact algorithms to find it. In this paper we analyze the bounds of path number in directed graph and we give certain properties characterizing directed acyclic graph that permit to give a structural representation of such graph. The algorithm used determines the topological ordering in time $O(n + |U|)$. We introduce two efficient algorithms that allow the construction of a minimal path-partition, one for the directed acyclic graph with time complexity $O(n^2 + n|U|)$ and the second for the strangely connected tournament having unique Hamiltonian circuit and having time complexity $O(n^2)$.

Keywords: Acyclic graph, Path Partition, tournament, Hamiltonian circuit, Adjacency list, Adjacency matrix, canonical ordering, Spanning tree.

1. Introduction

In management and economic, combinational problems necessitate a complicated formulation since their solution are not easily figured out, need complicated method and are sometime very difficult to set. The graph theory constitutes for instance, without any doubts, one of the most important and most efficient theories to model such kind of problem.

In fact we can use graph as tools to structure relationships among objects, variables etc… where the information can be represented in compact form. The concept of graph theory is therefore perfectly suitable to structure a problem in its initial analysis phase since a graph is the most general mathematical object. At the structural level (relational level) the nodes represent the objects, the variables etc…. and the arcs form the binary relation of influence among them.

Many real problems can be modeled as path-partition in directed graph that played particular role in the operation of arranging a set of points especially in case of directed acyclic graph (DAG). There exists several areas in which DAG arise as models e.g. project management, assignment problem network etc….we encounter such graphs in schedule problems, the analysis of language structure (Computation theory) the probability theory, the game theory etc… moreover managerial problem can be modeled as acyclic graphs. In the other hand the potential problem has a solution of certain type if and only if the graph G is acyclic see [17].

1.1. Concept of Graph

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The terminologies and notations are those of ([6], [8], [9], [12]). A directed graph is a pair $G = (X, U)$ where X is a finite set and U is a binary relation on X. The set X is called the vertex set and its elements are called vertices. The set U is called the arcs set and its element is called directed edges or arcs. A path is a sequence of vertices $(x_1, x_2, \dots, x_t)$ such that: $(x_i, x_{i+1}) \in U$ for $i = 1, 2, \dots., t-1$. The length of path is the number of arcs in the path. A path $C = (x_0, x_1, \dots, x_n)$ forms a circuit if $x_0 = x_n$ and the path contains at least an arc. A directed graph with no circuit is called acyclic (DAG).

Let $G = (X, U)$ be a directed graph, for $x \in X$, we denote by $G - x$ the sub- graph obtained from G by deleting the vertex x and the adjacency arcs to it. The out-degree of vertex x denoted $d_G^+(x)$ is the number of arcs leaving it and in-degree of vertex x denoted $d_G^-(x)$, is the number of arcs entering it.

From now on we denote:

$$I_G^+(x) = \{y \in X: (x, y) \in U\}$$
$$I_G^-(x) = \{y \in X: (y, x) \in U\}$$

$$X_G^+ = \{x \in X: d_G^+(x) > d_G^-(x)\}$$
$$X_G^0 = \{x \in X: d_G^+(x) = d_G^-(x)\}$$

A path partition of directed graph $G = (X, U)$ is a set T of arc-disjoint paths such that every arc in U is include in exactly one path of T. path my start and end anywhere, and they may be of any length including 0.

A minimum path partition of G is a path partition of G that use a fewest possible number of paths. The path number of directed graph G, denoted $P(G)$.

Definition 1.2

An asymmetric graph is a directed graph such that (x, y) is an arc implies (y, x) is not an arc. A tournament of order n denoted $T_n = (X, U)$ is a complete asymmetric graph on n vertices see [5, 7, 21, 24]

Definition 1.3

Let $G = (X, U)$ be a directed graph of order n. If $P(G) = e(G) = \sum_{x \in X_G^+}\left(d_G^+(x) - d_G^-(x)\right)$. We say G has the property Q.

2. Results on the path number in directed graph.

The arc–disjoint paths in a graph has an important application in several areas and needs exact algorithms to find it. Alspach and Pullman [4] have conjectured that for any simple graph G of order n, $P(G) \leq [n^2/4]$, O Brian [22], proved this conjecture. From O.Ore [23], we have:

$$P(G) \geq e(G) = \sum_{x \in X_G^+}\left(d_G^+(x) - d_G^-(x)\right)$$

Thus for a directed graph G, we deduce that:

$$e(G) \leq P(G) \leq \left[\frac{n^2}{4}\right]$$

For a further detailed study of P(G), we refer also to ([1], [11]).

2.1. Path-Partition in Tournaments

Theorem 1.

Let $T_n = (X, U)$ be a tournament of order n then $P[T_n] \geq \left[\frac{(n+1)}{2}\right]$.

The number of arc in tournament T_n, is given by the following result:

$$\sum_{x\in X}\left(d_G^+(x) - d_G^-(x)\right) = 2|U| = n(n-1)$$

Then $|U| = \frac{n(n-1)}{2}$

The maximum number of arcs in any path partition is $(n - 1)$. Thus the minimum number of paths needed to cover every arc in T_n is $\frac{n}{2}$, since $P[T_n]$ is an integer, we must have:

$$P[T_n] \geq \left\lceil\frac{(n+1)}{2}\right\rceil$$

From the preceding result and for any tournament T_n, we deduce then:

$$\left\lceil\frac{(n+1)}{2}\right\rceil \leq P[T_n] \leq \left\lceil\frac{n^2}{4}\right\rceil$$

Thereafter, we study the tournament T_n having a unique Hamiltonian circuit. A characterization of T_n have been given by Douglas [13].

Theorem 2.

For $n \geq 5$, a tournament $T_n = (X, U)$ admits a unique Hamiltonian circuit C if and only if the following conditions hold:

(i) There exist a partition of vertices in $X = A \cup B \cup \{x, y\}$

Where $A = \{a_1, \dots, a_p\}, B = \{b_1, \dots, b_q\}$ and $A = \{x, a_1, \dots, a_p, y, b_1, \dots, b_q, x\}$

(ii) $d^+_{T_n}(x) = 1 = d^-_{T_n}(y)$

(iii) $(b_q, a_1) \in U$ and $(a_p, b_1) \in U$

(iv) For $j > i + 1, (a_j, a_i) \in U$ and $(b_i, b_j) \in U$

(v) If $i < j$ and $u \leq v$ and if $(a_i, b_u) \in U$ then: $(a_j, b_v) \in U$

If $B = \emptyset$ we have a tournament having a unique Hamiltonian circuit and has exactly $m - n + 1$ elementary circuits. This tournament will be denoted $A_n = (X, U)$. The path $\lambda = (x, a_1, \dots, a_p, y)$ is a spanning tree and the vertices $(x, a_1, \dots, a_p, y)$ constitute the canonical ordering of A_n, and in the following this sequence will be denoted $(x_1, x_2, \dots, x_n)$.

The tournament A_n is characterized by:

$$(x_i, x_j) \in U \text{ if and only if } j = i + 1 \text{ or } i > j + 1$$

A curious fact concerning the number $W(n)$ of tournament T_n having a unique Hamiltonian circuit is equal to $(2n - 6)^{th}$ Fibonacci number. Garray [15] shows that for $n \geq 6$ we have:

$$W(n) = 3W(n - 1) - W(n - 2).$$

Gutin G. [16] provides a characterization allowing to find the number of non isomorphic tournament for $n \geq 5$

Paths and circuits are fundamental sub-structure in tournaments see : [5], [10], [19], [25], [26].

There is a $O(n)$ algorithm for finding Hamiltonian circuit in a tournaments ([20]).

Theorem 3. *(I. Abdel Kader [1]).*

Let $T_n = (X, U)$ be a tournament having a unique Hamiltonian circuit, then $P(T_n) \leq [n^2/4] - 2$.

For a further detailed study of tournament having unique Hamiltonian circuit and their number, we refer to ([15], [16]).

Property 1 $P(A_n) = \left[\frac{n^2}{4}\right] - 2$

It is clear that $X^+_{A_n} = \{x_i : i = [n/2] + 1, \ldots\ldots, n\}$

$$e(A_n) = \sum_{x \in X^+_{A_n}} \left(d^+_{A_n}(x) - d^-_{A_n}(x)\right)$$

but $d^+_{A_n}(x_i) - d^-_{A_n}(x_i) = 2(i - 1) - (n - 1)$ $(i = [n/2] + 1, \ldots, n - 1)$

$$d^+_{A_n}(x_n) - d^-_{A_n}(x_n) = n - 3$$

Then $e(A_n) = \sum_{i=1}^{n-1} \left(2(i-1) - (n-1)\right) + n - 3 = \left[\frac{n^2}{4}\right] - 2$

As $P(A_n) \geq e(A_n)$ and $P(A_n) \leq [n^2/4] - 2$

We have then the equality.

From the above result, the upper bound of $P(T_n)$ is the best possible.

It is important to note that there exist tournaments having a unique Hamiltonian circuit which are not isomorphic to A_n and which satisfy the equality $P(T_n) = e(T_n) = \left[\frac{n^2}{4}\right] - 2$. An example of this is the tournament T_n for which $|B| = 1$.

Algorithm 1. Path Partition of Tournament $\mathbf{A_n = (X, U)}$

Initialize the number of vertices n_0 of the tournament A_{n_0}

Determine the canonical ordering $(x_1, x_2, \ldots, x_{n_0})$ of this tournament

Initialize $T = \{(x_3 x_4, x_1, x_2), (x_4, x_2, x_3, x_1)\}$

$n = 5$

while $n \leq n_0$

$P = \emptyset$

For $j = \left[\frac{n}{2}\right] + 1$ to $n - 2$

$\mu = (x_n, x_j, \ldots, t)$ for $\lambda = (x_j, \ldots, t) \in T$

$P = P \cup \{\mu\}$

$T = T - \{\lambda\}$

End for

$P = P\ \{(x_{n-1}, x_n, x_1)\}$

$T = T \cup P$

For $k = 2$ to $\left[\frac{n}{2}\right]$

$T = T \cup \{(x_n, x_k)\}$

End for

17 $n = n + 1$

18 end while

Theorem 4 .

The path partition in tournament A_n can be computed in $O(n^2)$ time.

Proof.

We not by C_i the cost of statement i $(1 \le i \le 18)$. The algorithm 1 is based on 2 consecutive For Loops. The running time of the first For Loop is less than or equal to $\frac{n}{2}(C_8 + C_9 + C_{10})$, whereas for the second For Loop the running time is $\frac{n}{2}C_{15}$. The While Loop implicates that each statement is executed n times. The running time T_n of algorithm 1 is the sum of the running time of each statement executed. Then the worst running time can be expressed as:

$$T_n \le n\left(C_1 + C_2 + \cdots + \frac{n}{2}(C_8 + C_9 + C_{10}) + C_{13} + \frac{n}{2}C_{15} + \cdots\right)$$

Thus $T(n) = O(n^2)$

Application 1

Consider the tournament A_5 having the spanning tree $\lambda = (x_1, x_2, x_3, x_4, x_5)$

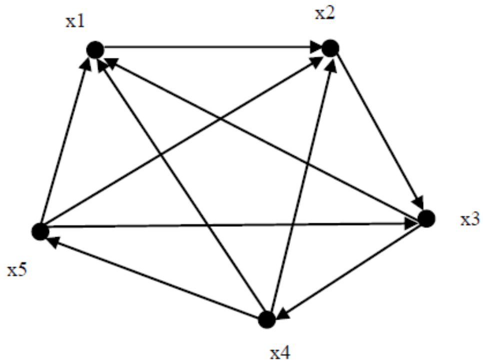

Figure 1. Spanning tree

Let $T_1 = \{(x_3x_4, x_1, x_2), (x_4, x_2, x_3, x_1)\}$ the set of arc-disjoint paths partitioning the arc of tournament A_4 generated by $X - \{x_5\}$. From T_1 we have the set of paths $T = \{(x_4, x_2, x_3, x_1), (x_4, x_5, x_1), (x_5, x_3x_4, x_1, x_2), (x_5, x_2)\}$ that partition the arcs of A_5.

2.2. Path Partition in Directed Acyclic Graph

In this section we prove that any directed acyclic graph G satisfies $P(G) = e(G)$. This result confirms that the lower bound of $P(G)$ is the best possible. The result obtained in this section will be interesting since there exist several important application areas in which directed acyclic graph arises as model: project management, assignment problem, network, etc… see Abdel Kader [2].

In the following we give a short and neat method to take advantage of directed acyclic graphs

Lemma 1.

Let $G = (X, U)$ be a graph of order n, if X included in $\Gamma_G^+(X)$, then G has at least a circuit.

Proof.

Let x_0 be a vertex of graph G, by hypothesis we have $x_0 \in \cup_{y \in X} \Gamma_G^+(y)$, then there exists at least an element $x_1 \in X$ with $x_0 \in \Gamma_G^+(x_1)\big((x_1, x_0) \in U\big)$ etc… at step $i \geq 1, x_{i-1} \in \Gamma_G^+(X)$, there exists a vertex $x_i \in X$ with $(x_i, x_{i-1}) \in U$, if x_i is a vertex already encountered we have a circuit if not the process continues and as X is finite, we will have a circuit.

Theorem 5.

Let $G = (X, U)$ be an acyclic graph, the following statements are held:

i. There exists at least a vertex x such that $d_G^-(x) = 0$
ii. The vertices of G can be arrayed in such a way the index of each vertex is less than the index of its successors.

Proof.

i. Assume $d_G^-(x) \geq 1$, for all $x \in X$, then there exists at least a vertex $x_1 \in X$ such that $x_{i-1} \in \Gamma_G^+(X)$ (for all $x \in X$), so X is included in $\Gamma_G^+(X)$, and from lemma 1, G has a circuit, contradiction then our assumption is false and there exists at least an $x \in X$, with $d_G^-(x) = 0$
ii. Let $A = \{x_{0i} \in X: d_G^-(x_{0i}) = 0\}$ and $G_1 = G_{X-A}$ the sub-graph of G induced by $X - A$. The graph G_1 is acyclic then there exists at least a vertex $x_2 \in X - A$ such that: $(x_{0k}, x_2) \in U$ for certain $x_{0k} \in A$, and so on, at step i we have the sub-graph $G_i = G_{i-1} - x_{i-1}$ of graph G. G_i is acyclic then there exists at least a vertex x_i such that$d_G^-(x_i) = 0$ and $(x_{i-1}, x_i) \in U$. Thus the vertices of G can be arrayed in such a way that the index of each vertex is less than the index of its successors.
It is obvious that the condition (ii) is equivalent to the fact that G is directed acyclic graph.

From the previous result we deduce that the vertices of the directed acyclic graph G can be indexed as: $x_1, x_2, \dots, x_n$ such that: $d_G^+(x_1) \geq d_G^+(x_2) \dots \geq d_G^+(x_n)$ where the arcs in G run from left to right. In this way we have a topological ordering of graph G.

Theorem 6. The topological ordering can be computed in $O(n + |U|)$ time.

To prove this result, enumerate the arcs of G one by one, this allows the computation of the in-degree $\big(d_G^-(x)\big)$ for all node i in linear time. Consider the array L that contains all the sources of graph G. Now execute the following algorithm, using an auxiliary list L' that is initially empty:

```
Procedure:    Typological Ordering (G)
              repeat
              for each vertex i ∈ L do
                      for each arc (i, j) ∈ U do
                              begin
                              d⁻G(j) = d⁻G(j) − 1
                              If d⁻G(j) = 0 then add j∈ L'
                              end
                      print L
                      L = L'
                      L' = ∅
                      until L = ∅
```

It is obvious that the computation takes only $O(n + |U|)$ total time since every node and every arc appears precisely one in the process.

Theorem 7.

Let $G = (X, U)$ be a directed graph of order n and $v \in X_G^+$. If G satisfies the following conditions:

i. $d_G^-(v) = 0$
ii. $P(G - v) = e(G - v)$ (that is $G - v$ has the property Q),

Then G has the property Q.

Proof.

$X_{G-v}^+ = (X_G^+ - \{v\}) \cup \left(X_G^+ \cap \Gamma_G^+(v)\right)$, and $P(G - v) = e(G - v)$. If $x \in X_{G-v}^+ \cap \Gamma_G^+(v)$, then $d_{G-v}^+(x) = d_G^+(x)$ and $d_{G-v}^-(x) = d_G^-(x) - 1$. Moreover, for $x \in X_{G-v}^+ - \left(X_{G-v}^+ \cap \Gamma_G^+(v)\right)$, we have: $d_{G-v}^+(x) = d_G^+(x)$ and $d_{G-v}^-(x) = d_G^-(x)$.

But in $G - v$, through each vertex $x \in X_{G-v}^+ \cap \Gamma_G^+(v)$ there pass $\left(d_{G-v}(x) - d_{G-v}^-(x)\right) + 1$ elementary paths of origin x which belong to a path partition R of the arcs of the digraph, the cardinal of R being $P\left(G - v\right)$. Among those paths of origin x, consider the path $\lambda = (x, \dots)$. Since $(v, x) \in U$, the path λ allows the construction in G of the path $\mu = (v, x, \dots)$ of origin v. Thus the number of paths of origin x in G becomes$\left(d_G^+(x) - d_G^-(x)\right)$. Moreover, for each $x \in \Gamma_G^+(v) - \left(X_{G-v}^+ \cap \Gamma_G^+(v)\right)$, we construct the path (v, x) of origin v in G. Let R' be the set of elementary paths obtained from R by cancelling those paths λ which have been used to define the path μ of origin v in G. Let T be the following set of elementary paths:

$$T = R' \cup \{\mu = (v, x, \dots) \mid x \in X_{G-v}^+ \cap \Gamma_G^+(v)\} \cup \left\{(v, x) \mid x \in \Gamma_G^+(v) - \left(X_{G-v}^+ \cap \Gamma_G^+(v)\right)\right\}$$

It is obvious that the set T partitions the arcs of G, and we have $|T| \leq e(G) \leq P(G)$.

Therefore $P(G) = e(G)$.

Remark. If we replace the condition (i) of the Theorem by condition (i'), $d_G^+(x) = 0$, we get a similar result. Moreover the preceding Theorem allows us the construct from a digraph of order $\left(n - 1\right)$ satisfying Q, another digraph of order n still satisfying Q.

From this theorem, we deduce the following results:

Property 2

Let $G = (X, U)$ be a directed acyclic graph, then:

$$P(G) = e(G)$$

Proof.

By recurrence on the number n of vertices, for $n = 3, 4$ the property is true. Assume that it is true for any acyclic graph with $n - 1$ vertices and prove it for any acyclic graph having n vertices. In G there exists a vertex v such that $d_G^-(x) = 0$. The sub graph G_1 of G induced by $X - v$ is an acyclic graph with $n - 1$ vertices, then from our assumption $P\left(G - v\right) = e\left(G - v\right)$. Thus from theorem 7 we have $P(G) = e(G)$.

Remark: If TT_n is the transitive tournament then the vertices of TT_n can be arrayed as:

$X = \{x_1, x_2, \dots, x_n\}$ where: $d_{TT_n}^+(x_i) = n - i$ for $i = 1, 2, \dots n$.

Property 3

If TT_n, is the transitive tournament of order n, then $P(TT_n) = [n^2/4]$.

TT_n is an acyclic tournament then from Property 2 we have:

$$P(TT_n) = e(TT_n) = (n - 1) + (n - 3) + \dots + 1 = [n^2/4].$$

This result prove that the upper bound of $P(G)$ is the best possible.

The following algorithm allows finding a path partition in directed acyclic graph $G = (X, U)$.

Algorithm 2. Path Partition $(\mathbf{G})$

1 Call Topological ordering (G)

2 Determine the set S of sinks of graph G

3 $i = n - |S|$

4 Initialize $T = \{x_n, x_{n-1}, x_{n-|S|+1}\}$

5 While $i \geq 1$

6 $P = \emptyset$

7 While $x \in \Gamma_G^+(x_i)$

8 If $\lambda = (x, \dots, t) \in T$ then

9 $\mu = (x_i, x, \dots, t)$

10 $T = (T - \{\lambda\}) \cup \{\mu\}$

11 Else $P = P \cup \{(x_i, x)\}$

12 End if

13 $T = T \cup P$

14 $i = i - 1$

15 end while

Theorem 8.

The path partition in directed acyclic graph can be computed in $O(n^2)$ time.

Proof.

We not by C_i the cost of the statement i for $1 \leq i \leq 15$. The statement 1 can be executed in $T(1) = C_1(n + |U|)$ time but the maximal value of $|U|$ is around $n^2 (|U| \approx n^2)$. Then $T(1) \leq C_1 n^2$. The algorithm 2 is based on 2 nested While Loops. In the worst case the internal While Loops has a running time of $n(C_8 + C_9 + \cdots + C_{13})$. Since the running time of algorithm 2 is the sum of the running time of each statement executed; from the external While Loop of algorithm 2, the worst running time $T(n)$ can be expressed as:

$$T(n) \leq C_1 n^2 + \cdots + C_7 + n\,(n(C_8 + C_9 + \cdots + C_{13})) + \cdots$$

$$T(n) \leq C_1 n^2 + K n^2 \leq C n^2$$

Thus $T(n) = O(n^2)$

Application 2

Consider the following acyclic graph $G = (X, U)$

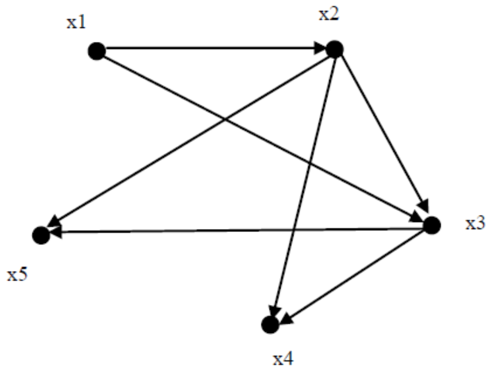

Figure 2. Acyclic Graph $G = (X, U)$

The topological ordering of graph G is $\{x_1, x_2, x_3, x_4, x_5\}$

Let $T_1 = \{(x_2, x_3, x_4), (x_3, x_5), (x_2, x_4), (x_2, x_5)\}$ be a set of arc-disjoint paths partitioning $G_1 = G - x_1$. From T_1 we have:

$$T = \{(x_1, x_2, x_3, x_4), (x_1, x_3, x_5)(x_2, x_4), (x_2, x_5)\}.$$

2.3. Computer Representation

The particular implementation for the graph G, can have a profound effect on the complexity of algorithm. In the following we give the most useful representation, for more details we refer to: [3], [14], [18].

2.3.1. Vertex Query Representation

The first representation use the Adjacency matrix $A(G) = A(n, n)$ of graph G is define as follows:

$$A(i, j) = 1 \text{ if and only if } (i, j) \in U \text{ and } 0 \text{ otherwise.}$$

The adjacency matrix requires $O(n^2)$ storage locations while retaining $O(1)$ time access to its elements.

We note that the form of adjacency matrix $A(G)$, depends on the order in which the vertices of G can be arrayed. Then we have the following result:

Theorem 9.

Two graphs G and G' are isomorphic if and only if $A(G) = A(G')$.

Proof.

If G and G' are isomorphic then $(x_i, x_j) \in U$ if and only if $\left(f(x_i), f(x_j)\right) = (y_i, y_j) \in U'$ if and only if $A(i, j) = 1$ and $A'(i, j) = 1$. Then $A(G) = A(G')$

If $A(G) = A(G')$, it is obvious that G and G' are isomorphic.

We deduce then that the order in which the adjacency matrix is written does not have any influence on the result of computation.

2.3.2. Adjacency list representations

The adjacency list of graph $G = (X, U)$ consists of an array Adj of $|X|$ lists, one of each vertex $x \in X$. Adj (x) contains all the vertices $y \in X$ such that: $(x, y) \in U$. We note that in Adj (x) the vertices are stored in any arbitrary order and are usually a more compact representation that the adjacency matrix. The sum of the length of the entire adjacency list is $|U|$. The adjacency list representation requires $O(n + |U|)$ storage locations.

The simplicity of a adjacency matrix may make it preferable when graphs are reasonable small.

If G has a particular representation, it may well be exploited to give a suitable representation in computer storage ([18]). For example if the graph $G = (X, U)$ is acyclic, from the above theorem the nodes of G can be arrayed in such a way that all arc run strictly from left to right; we obtain then the topological ordering of graph G.

Conclusion

There exists theoretical and practical reasons for studying special classes of directed graph, and it can be very interesting and worthwhile to explore a graph – algorithm problem for special types of graphs. This leads to study the very important class, we mean the networks, related to important problems in several field such as: Science, Economic, Management, Wireless network etc….. Many real problems can be modeled as path partition problems in directed graph especially in the case of network. The problem of constructing of arc-disjoint paths is a hard problem and has been studied by several authors.

In this paper, we give some properties concerning the number $P(G)$ of arc-disjoint paths partitioning the arcs of a given graph G, also certain properties characterizing the directed acyclic graph, that allow to give a structural representation of such class of graph. The result obtained facilitates the implementation of the given algorithms, so the problem of finding minimum arc-disjoint paths is of obvious interest in many network problems.

For future, research, we plan to study the most important areas related to the problem of arc-disjoint paths such as: schedule problems, network etc… especially the wireless sensor network.

In sensor network, to manage how a sensor node uses its power, we need a power management plan that allows the sensor nodes to work together in a power efficiency way, to transfer data in wireless network. In other words the goal is to extend the life time of the network by reducing the every use in the routing phase while maintaining a similar level of resilience to node failures. To achieve this objective, we need a routing protocol for energy efficiency in real-time communication over sensor network, avoiding then each sensor to work on its own.

The result obtained from these algorithms can be used to provide a reliable transmission of the entire data sent from the source to the sink over the available disjoint paths, which will be split into sub-packets corresponding to the number of available paths to ensure efficient energy consumption. Actually we work in the implementation of a new method to resolve such kind of problems.

We propose the following conjecture**:**

For any strongly connected graph G of order $\mathbf{n}, \mathbf{P(G)} \neq \left[\frac{\mathbf{n^2}}{\mathbf{4}}\right]$**.**

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Parallel Surfaces of Spacelike Ruled Weingarten Surfaces in Minkowski 3-space

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Abstract: In this work, it is shown that parallel surfaces of spacelike ruled surfaces which are developable are spacelike ruled Weingarten surfaces. It is also shown that parallel surfaces of non-developable ruled Weingarten surfaces are again Weingarten surfaces. Finally, some properties of that kind parallel surfaces are obtained in Minkowski 3-space.

Keywords: Spacelike Weingarten surface, parallel surface, spacelike ruled Weingarten surface, curvatures

1. Introduction

A Weingarten (or W-) surface is a surface on which there exists a relationship between the principal curvatures. Let f and g be smooth functions on a surface M in Minkowski 3-space. The Jacobi function $\Phi(f,g)$ formed with f,g is defined by $\Phi(f,g) = det\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ where $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$. In particular, a surface satisfying the Jacobi condition $\Phi(K,H) = 0$ with respect to the Gaussian curvature K and the mean curvature H is called a Weingarten surface or W-surface. All developable surfaces $(K = 0)$ and minimal surfaces $(H = 0)$ are Weingarten surfaces. Some geometers have studied Weingarten surfaces and obtained many interesting results in both Euclidean and Minkowskian spaces [1,2,3,4,5,6,7,11,12,13,19,20].

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite number of surfaces because we choose the constant distance along the normal, arbitrarily. From the definition, it follows that a parallel surface can be regarded as the locus of point which are on the normals to M at a non-zero constant distance r from M.

In this paper, we study on parallel surfaces of spacelike surfaces which become both ruled and Weingarten surfaces. We show that parallel surface of a ruled Weingarten surface is again a Weingarten surface in Minkowski 3-space. Also, some properties of that kind parallel surfaces are given in Minkowski 3-space.

2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$\langle dx, dx\rangle = dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in R^3. An arbitrary vector x of $\mathbb{E}_1^3$ is said to be spacelike if $\langle x, x\rangle > 0$ or $x = 0$, timelike if $\langle x, x\rangle < 0$ and lightlike or null if $\langle x, x\rangle = 0$ and $x \neq 0$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $x \in \mathbb{E}_1^3$ the norm is defined by $\|x\| = \sqrt{\langle x, x\rangle}$, then the vector x is called a spacelike unit vector if $\langle x, x\rangle = 1$ and a timelike unit vector if $\langle x, x\rangle = -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [15]. For any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $x \times y = \big((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1)\big)$ [14].

A surface in the Minkowski 3-space $\mathbb{E}_1^3$ is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric. This is equivalent to saying that the normal vector on the spacelike surface is a timelike vector [16].

A (differentiable) one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns each $u \in I$ to a point $\alpha(u) \in \mathbb{E}_1^3$ and a vector $X(u) \in \mathbb{E}_1^3, X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ depend differentiable on u. For each $u \in I$, the line L_u which passes through $\alpha(u)$ and parallel to $X(u)$ is called the *line of the family* at u.

Given a one-parameter family of lines $\{\alpha(u), X(u)\}$, the parametrized surface

$$\varphi(u, v) = \alpha(u) + vX(u), \qquad u \in I, \quad v \in \mathbb{R} \tag{2.1}$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$. The lines L_u are called the *rulings* and the curve $\alpha(u)$ is called a *directrix* of the surface φ. The normal vector of surface is denoted by $\vec{N}$ [16].

Theorem 2.1. Using standard parameters, a ruled surface in Minkowski 3-space is up to Lorentzian motions, uniquely determined by the following quantities:

$$Q = \langle \alpha', X \wedge X'\rangle, J = \langle X, X'' \wedge X'\rangle, F = \langle \alpha', X\rangle \tag{2.2}$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [13].

Theorem 2.2. The Gauss and mean curvatures of spacelike ruled surface φ in terms of the parameters Q, J, F, D in $\mathbb{E}_1^3$ are obtained as follows:

$$K = \frac{Q^2}{D^4} \text{ and } H = \frac{1}{2D^3}[QF - Q^2J - vQ' - v^2J], \tag{2.3}$$

where $D = \sqrt{-\varepsilon Q^2 + \varepsilon v^2}$ [4].

Definition 2.3. A surface is called a Weingarten surface or W-surface in $\mathbb{E}_1^3$ if there is a nontrivial relation $\Phi(K, H) = 0$ or equivalently if the gradients of K and H are linearly dependent. In terms of the partial derivatives with respect to u and v, this is the equation

$$K_uH_v - K_v\,H_u = 0, \tag{2.4}$$

where K and H are Gaussian and mean curvatures of surface, respectively [4].

Theorem 2.4. The ruled surface φ is a Weingarten surface if and only if the invariants Q, J and F are constant in $\mathbb{E}_1^3$ [4].

Theorem 2.5. Parameter curves are lines of curvature if and only if $F = f = 0$ in $\mathbb{E}_1^3$ [14].

Lemma 2.6. A point **p** on a surface M in $\mathbb{E}_1^3$ is an umbilical point if and only if

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g} \tag{2.5}$$

[10].

Definition 2.7. Let M and M^r be two surfaces in Minkowski 3-space. The function

$f\colon M \longrightarrow M^r$

$p \longrightarrow f(p) = p + rN_p$

is called the parallelization function between M and M^r and furthermore M^r is called parallel surface to M in $\mathbb{E}_1^3$ where r is a given positive real number and **N** is the unit normal vector field on M [8].

The representation of points on M^r can be obtained by using the representations of points on M in Minkowski 3-space. Let φ be the position vector of a point P on M and φ^r be the position vector of a point $f(P)$ on the parallel surface M^r. Then $f(P)$ is at a constant distance r from P along the normal to the surface M. Therefore the parametrization for M^r is given by

$$\varphi^r(u,v) = \varphi(u,v) + r\mathbf{N}(u,v) \tag{2.6}$$

where r is a constant scalar and **N** is the unit normal vector field on M [17].

Theorem 2.8. Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f\colon M \to M^r$ be the parallelization function. Then for $X \in X(M)$,

1. $f_*(X) = X + rS(X)$
2. $S^r\big(f_*(X)\big) = S(X)$
3. f preserves principal directions of curvature, that is
$S^r\big(f_*(X)\big) = \dfrac{k}{1+rk} f_*(X)$

where S^r is the shape operator on M^r and k is a principal curvature of M at p in direction X [8].

Theorem 2.9. Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f\colon M \to M^r$ be the parallelization function. Then f preserves becoming umbilical point on the surface M^r in Minkowski 3-space [18].

Theorem 2.10. Let M be a spacelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$ Then we have

$$K^r = \frac{K}{1-2rH-r^2K} \text{ and } H^r = \frac{H+rK}{1-2rH-r^2K} \tag{2.7}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r respectively [17].

Corollary 2.11. Let M be a spacelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Then we have

$$K = \frac{K^r}{1-2rH^r-r^2K^r} \quad \text{and } \mathrm{H} = \frac{H^r+rK^r}{1-2rH^r-r^2K^r} \tag{2.8}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r respectively [17].

Theorem 2.12. . Let M be a spacelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Parallel surface of a spacelike developable ruled surface is again a spacelike ruled surface [17].

Theorem 2.13. Let $\varphi(u,v)$ be a spacelike surface in $\mathbb{E}_1^3$ with $F = f = 0$. Then the parallel surface

$$\varphi^r(u,v) = \varphi(u,v) + rN(u,v)$$

is a developable ruled surface while one of the parameters of parallel surface is constant and the other is variable [17].

Corollary 2.14. Let M be a spacelike ruled surface and M^r be a spacelike parallel surface of M in $\mathbb{E}_1^3$. The Gaussian and mean curvatures, respectively, K^r and H^r are as follows:

$$K^r = \frac{Q^2}{D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2} \tag{2.9}$$

$$H^r = \frac{QFD - Q^2JD - vQ'D - v^2JD + 2rQ^2}{2D^4 - 2rQFD + 2rQ^2JD + 2rvQ'D + 2rv^2JD - 2r^2Q^2} \tag{2.10}$$

in terms of the parameters Q, J, F, D [17].

3. Parallel surfaces of spacelike ruled Weingarten surfaces in $\mathbb{E}_1^3$

Let M^r be a parallel surface to a surface M in Minkowski 3-space. If there is a nontrivial relation as

$$\Phi(K^r, H^r) = 0 \tag{3.1}$$

between the Gaussian curvature K^r and the mean curvature H^r of the parallel surface M^r, the parallel surface M^r is said to be Weingarten surface as in analog to the original surface. In other words, if jacobi determinant as a relation between the Gaussian curvature K^r and the mean curvature H^r of the parallel surface M^r vanishes, the following condition for parallel Weingarten surfaces

$$\Phi(K^r, H^r) = \det\begin{pmatrix} K_u^r & K_v^r \\ H_u^r & H_v^r \end{pmatrix} = K_u^r H_v^r - K_v^r H_u^r = 0 \tag{3.2}$$

is obtained.

Theorem 3.1. Let M be a developable spacelike ruled surface in $\mathbb{E}_1^3$, then parallel surface M^r of M is a spacelike parallel ruled Weingarten surface.

Proof. From Theorem 2.11, Parallel surface of developable spacelike ruled surface M is again a developable spacelike ruled surface. Therefore, Gaussian curvature of parallel surface K^r vanishes since $K = 0$ for M. It means that the surface is a Weingarten surface.

Theorem 3.2. Let $\varphi(u, v)$ be a spacelike surface in $\mathbb{E}_1^3$ with $F = f = 0$. Then the parallel surface

$$\varphi^r(u, v) = \varphi(u, v) + rN(u, v)$$

is a ruled Weingarten surface while one of the parameters of parallel surface is constant and the other is variable.

Proof. The surface $\varphi^r(u, v)$ is a developable spacelike ruled surface from Theorem 2.12, hence K^r vanishes by putting $K = 0$ in Theorem 2.10. Consequently, the surface is also Weingarten surface.

Theorem 3.3. Let φ^r be a parallel surface of a spacelike ruled surface φ in Minkowski 3-space. If φ is a Weingarten surface if and only if φ^r is a Weingarten surface.

Proof. ($\Longrightarrow$)**:** If φ is a spacelike ruled surface in $\mathbb{E}_1^3$, then we have to show the equation (3.2) by using the equation (2.4). First, using the expressions of (2.7) in (3.2), we have

$$\begin{aligned} K_u^r H_v^r - K_v^r H_u^r &= \left(\frac{K}{1 - 2rH - r^2K}\right)_u \left(\frac{H + rK}{1 - 2rH - r^2K}\right)_v - \left(\frac{K}{1 - 2rH - r^2K}\right)_v \left(\frac{H + rK}{1 - 2rH - r^2K}\right)_u \\ &= \Phi\{[K_u - 2rK_uH + 2rKH_u].[H_v + rK_v - r^2K_vH + r^2KH_v] \\ &\quad - [K_v - 2rK_vH + 2rKH_v].[H_u + rK_u - r^2K_uH + r^2KH_u]\} \end{aligned} \tag{3.3}$$

where $\Phi = \frac{1}{(1-2rH-r^2K)^4}$. If we make computations in (3.3), we get

$$\begin{aligned} K_u^r H_v^r - K_v^r H_u^r = \Phi\{&K_uH_v + rK_uK_v - r^2K_uK_vH + r^2KK_uH_v - 2rHK_uH_v - 2r^2K_uK_vH + \\ &2r3KuKvH2 - 2r3KHKuHv + 2rKHuHv + 2r2KKvHu - 2r3KHKvHu + 2r3K2HuHv + 2rHKvHu + r2 \\ &HKuKv - r2KKvHu - KvHu - 2r3K2HuHv + 2r2HKuKv - 2r3H2KuKv + 2r3KHKvHu - 2rKHuHv - r \\ &2KKuHv + 2r3KHKuHv - rKuKv. \end{aligned} \tag{3.4}$$

After making arrangements in (3.4), the equation becomes as

$$K_u^r H_v^r - K_v^r H_u^r = \frac{1}{(1-2rH-r^2K)^4}\{[K_uH_v - K_vH_u] - 2r[K_uH_v - K_vH_u] - r^2K[K_uH_v - K_vH_u]\}. \tag{3.5}$$

Later, using (2.4) in (3.5),

$$K_u^r H_v^r - K_v^r H_u^r = 0 \tag{3.6}$$

is found. Parallel surface φ^r is a Weingarten surface since (3.6).

(⟸): Conversely, let φ^r be a Weingarten surface which is parallel to a spacelike ruled surface, then it satisfies (3.2). Hence, the equation (2.4) has to be shown. First, using the expressions of (2.8) in (2.4), we have

$$K_uH_v - K_vH_u = \left(\frac{K^r}{1+2rH^r-r^2K^r}\right)_u \left(\frac{H^r+rK^r}{1+2rH^r-r^2K^r}\right)_v - \left(\frac{K^r}{1+2rH^r-r^2K^r}\right)_v \left(\frac{H^r+rK^r}{1+2rH^r-r^2K^r}\right)_u = \Psi\{[K_u^r + \\ 2rKurHr-2rKrHur.Hvr-rKvr-r2KvrHr+r2KrHvr-Kvr+2rKvrHr-2rKrHvr.Hur-rKur-r2 \\ KurHr+r2KrHur \tag{3.7}$$

where $\Psi = \frac{1}{(1-2rH^r-r^2K^r)^4}$ If we make computations in (3.6), we get

$$K_uH_v - K_vH_u = \Psi\{K_u^r H_v^r - rK_v^r K_u^r - r^2H^rK_u^rK_v^r + r^2K^rK_u^rH_v^r + 2rH^rK_u^rH_v^r - 2r^2H^rK_u^rK_v^r - \\ 2r3Hr2KurKvr+2r3KrHrKurHvr-2rKrHurHvr+2r2KrKvrHur+2r3KrHrKvrHur-2r3Kr2H \\ urHvr-KvrHur+rKurKvr+r2HrKurKvr-r2KrKvrHur-2rHrKvrHur+2r2HrKurHvr+2r3Hr \\ 2KurKvr-2r3KrHrKvrHur+2rKrHurHVr-2r2KrKurHvr-2r3KrHrKurHvr+2r3Kr2HurHvr. \tag{3.8}$$

Making arrangements in (3.8), we get the following equation:

$$K_uH_v - K_vH_u = \frac{1}{(1+2rH^r-r^2H^r)^4}\{(K_u^rH_v^r - K_v^rH_u^r) + r^2K^r(K_u^rH_v^r - K_v^rH_u^r) + 2rH^r(K_u^rH_v^r - K_v^rH_u^r) + \\ 2r2Kr(KurHvr-KvrHur). \tag{3.9}$$

By using (3.2) in (3.9)

$$K_uH_v - K_vH_u = 0 \tag{3.10}$$

is found. Since (3.6), spacelike ruled surface φ is a Weingarten surface.

Corollary 3.4. The surface φ^r which is parallel to spacelike ruled surface φ is aWeingarten surface if and only if the invariants Q, J, F which determine spacelike ruled surface φ are constant.

Proof. (⟹)**:** Let parallel surface φ^r be a Weingarten surface. Then from Theorem 3.3, spacelike ruled surface φ is also a Weingarten surface. Hence, the invariants Q, J, F are constants from Theorem 2.4.

(⟸): Let the invariants Q, J, F be constants, then spacelike ruled surface φ is a Weingarten surface from Theorem 2.4. The parallel surface φ^r is also a Weingrten surface from Theorem 3.3.

Corollary 3.5. Let the surfaces φ and φ^r be, respectively, spacelike ruled Weingarten surface and its parallel surface in $\mathbb{E}_1^3$. Then, there is the relation

$$H^r = \left(r + \frac{H}{K}\right)K^r \tag{3.11}$$

among Gauss K^r and mean H^r curvatures of spacelike parallel Weingarten surface φ^r and Gauss K and mean H curvatures of spacelike ruled Weingarten surface φ.

Proof. We will use the values of the curvatures K^r and H^r given in (2.9) and (2.10). Let

$$A = D^4 - rQFD + rQ^2JD + rvQ^{'}D + rv^2JD - r^2Q^2. \tag{3.12}$$

By using (3.12) in (2.9), we get

$$A = \frac{Q^2}{K^r} \tag{3.13}$$

or

$$A = \frac{QFD - Q^2JD - vQ^{'}D - v^2JD + 2rQ^2}{2H^r} \tag{3.14}$$

is found. From (3.13) and (3.14),

$$H^rQ^2 = [(QF - Q^2J - vQ^{'} - v^2J)D + 2rQ^2]K^2 \tag{3.15}$$

is obtained. By using the expressions of Theorem 2.2 in (3.15), we have

$$H^r = \frac{2D^4H + 2rQ^2}{2Q^2}K^r \tag{3.16}$$

By using the expressions of Theorem 2.2 in (3.16), we get

$$H^r = \left(r + \frac{H}{K}\right)K^r.$$

Lemma 3.6. Let φ be a non-developable spacelike ruled surface and φ^r be parallel surface of φ in $\mathbb{E}_1^3$. Then there is no umbilical point on spacelike parallel Weingarten surface φ^r.

Proof. Let spacelike ruled Weingarten surface φ be given as the following parameterization:

$$\varphi(u,v) = \alpha(u) + vX(u),\ \ \langle\alpha^{'},\alpha'\rangle = 1, \langle X,X\rangle = 1, \langle X^{'},X'\rangle = \varepsilon \tag{3.17}$$

where $\varepsilon = \pm 1$ in $\mathbb{E}_1^3$. From Theorem 2.4, the invariants Q, J, F have to be constant for ruled surface to become Weingarten surface. If there is a umbilical point on spacelike ruled Weingarten surface, from Lemma 2.6, it has to be

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g}. \tag{3.18}$$

Coefficients of first fundamental I for the surface φ are as follows:

$$E = \langle\varphi_u,\varphi_u\rangle = 1 + \ \varepsilon v^2, F = \langle\varphi_u,\varphi_v\rangle = \langle\alpha^{'},X'\rangle, G = \langle\varphi_v,\varphi_v\rangle = 1. \tag{3.19}$$

Thereby normal vector of the surface φ is $N = \alpha^{'} \wedge X + vX' \wedge X$. Coefficients of second fundamental II for the surface φ are obtained as

$$\begin{aligned} e = \langle\varphi_{uu},N\rangle = \langle\alpha^{''},\alpha^{'} \wedge X\rangle + v\langle\alpha^{''},X^{'} \wedge X\rangle + v\langle X^{''},\alpha^{'} \wedge X\rangle + v^2\langle X^{''},X^{'} \wedge X\rangle \\ f = \langle\varphi_{uu},N\rangle = \langle\alpha^{'},X \wedge X^{'}\rangle \end{aligned} \tag{3.20}$$

and

$$g = \langle\varphi_{uu},N\rangle = 0 \tag{3.21}$$

By using (3.19), (3.20) and (3.21) in (2.5), we have

$$Fg - Gf = -\langle X,X\rangle\langle\alpha^{'},X \wedge X^{'}\rangle. \tag{3.22}$$

The equation (3.22) means that there is no umbilical point on a non-developable spacelike ruled surface φ. Hence there is also no umbilical point on parallel surface φ^r of φ from Theorem 2.9.

Example 3.7. Let's take helicoidal surface given as the following parameterization:

$$\varphi(u,v) = (v\sinh u\,, v\cosh u\,, u). \tag{3.23}$$

This surface is a spacelike surface for the interval $-1 < v < 1$. If we compute the values of Q, J and F in Theorem 2.2 for that surface, then

$$Q = -1, J = 0 \text{ and } F = 0 \tag{3.24}$$

are obtained. The invariants Q, J and F in (3.23) are constant therefore, from Corollary 3.5, parallel surface φ^r is a Weingarten surface. The parametric equation of φ^r which is parallel to spacelike ruled Weingarten surface φ given in (3.23) is obtained as follows

$$\varphi^r(u,v) = \left(v\sinh u + \frac{r\cosh u}{\sqrt{1-v^2}}, v\cosh u + \frac{r\sinh u}{\sqrt{1-v^2}}, u + \frac{vr}{\sqrt{1-v^2}}\right). \tag{3.25}$$

The figures 3.1 and 3.2 show, respectively, spacelike ruled Weingarten surface φ given in (3.23) and its parallel together with the original surface. The blue surface in Figure 3.2 show parallel surface, and the red one is for the original surface.

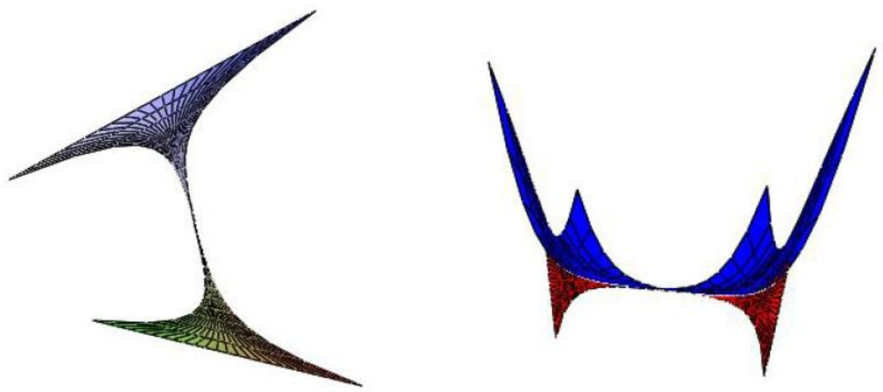

Figure 3.1 Figure 3.2

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The chaotic behaviour on transition points between parabolic orbits

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Abstract: The potential energy surfaces interact each other and their curvilinear coordinates have the critical information about disturbance at interaction points. Therefore, transition points between parabolic orbits that are solutions of one differential equation with variable coefficients is studied in this paper. Also we present an approach for the chaotic behaviour on transition points of the parabolic orbits.

Keywords: Chaotic Behaviour, Differential Equation, Parabolic Intersections, Quadratic Function, Schwarzian Derivative,

1. Introduction

A dynamical system determines the present state in terms of past states and is described by equations representing the evolution of a solution with time, initial conditions and control signals [1]. The Intersections of parabolic orbits play an important role in analysis, design and synthesis of control laws in dynamical systems. In an elliptical or circular orbit, the planet is moving slower than the escape velocity. One focus of the ellipse is on or near the primary and the second focus is on an unoccupied point in space. A planet in this type of orbit will continue to orbit the primary unless another force alters its path. All planets and moons in solar system follow this type of orbit. In a parabolic orbit, the planet has reached escape velocity. A planet with this type of orbit will never return to orbit the primary again. Also in a hyperbolic orbit, which is somewhat more flattened than a parabolic orbit, the planet is moving faster than escape velocity. A satellite leaving earth's orbit will follow parabolic or hyperbolic path [2-8].

The interactions points on intersection from a parabola orbit to other will be computed in this study. A parabola is quadratic function. If the quadratic equations are the solutions of the same differential equation with variable coefficients, the points of intersection between orbits give knowledge about the interactions and transitions. Furthermore, these equations have two distinct real roots. Hence, the chaotic behaviour of the transition between the points of parabolic intersections will be studied by using the methods of Schwarzian Derivative, Lyapunov Exponent and Bifurcation Diagram [9-16].

2. The parabolas which are the solutions of differential equation with variable coefficients

A quadratic function is $f(x) = ax^2 + bx + c$, where x is path-dependent variable and $a, b,$ and c are constants. In this study, we assume that, $a > 0$, c only changes the vertical position. The quadratic equations which have distinct real roots, open upward and downward structure are given (1) and (2).

$$f_1(x) = ax^2 - (a+c)x + c \tag{1}$$

$$f_2(x) = -ax^2 + (a-c)x + c \tag{2}$$

We assume that (1) and (2) are the solutions of second order differential equation with variable coefficients as given below,

$$f''(x) + p(x)f(x) + q(x) = 0 \tag{3}$$

We have to verify that if $f_1(x)$ and $f_2(x)$ are the solutions of equation in (3). Further, to determine the coefficients of $p(x)$ and $q(x)$, we substitute $f_1(x)$, $f_2(x)$, $f_1''(x)$ and $f_2''(x)$ into the differential equations in (3), and eventually obtain (4) and (5)

$$f_1''(x) + p(x)f_1(x) + q(x) = 0 \tag{4}$$

and

$$f_2''(x) + p(x)f_2(x) + q(x) = 0 \tag{5}$$

Where $f_1''(x)$ and $f_2''(x)$ are the second order derivatives of the $f_1(x)$ and $f_2(x)$ respectively. $f_1''(x) = 2a$, $f_1(x)$, quadratic function opens upward and $f_2''(x) = -2a$, $f_2(x)$, quadratic function opens downward (a>0). Variable coefficients $p(x)$ and $q(x)$ are obtained from (4) and (5),

$$p(x) = -\frac{2}{x^2 - x}, \tag{6}$$

$$q(x) = -\frac{2c}{x}, \tag{7}$$

$$c = 1. \tag{8}$$

If we substitute $p(x)$, $q(x)$ and c in to the differential equation with variable coefficients as given (3), the differantial equation is therefore,

$$\frac{d^2 f(x)}{dx^2} - \frac{2}{x^2 - x} f(x) = \frac{2}{x} \tag{9}$$

The solutions of the differential equation in (9) are found as (10) and (11).

$$f_1(x) = ax^2 - (a+1)x + 1 \tag{10}$$

$$f_2(x) = -ax^2 + (a-1)x + 1 \tag{11}$$

The roots of the parabola equation $f_1(x)$ are $\left(1, \frac{1}{a}\right)$, and the roots of the parabola equation $f_2(x)$ are $\left(1, -\frac{1}{a}\right)$, where a>0. As shown in Fig.1, the transition points of parabolas affect each other.

3. Chaotic behaviour on the transition points of parabolas

3.1. Schwarzian derivative

Schwarzian derivative is an important criterion for the chaotic behaviour of discrete dynamic systems [9,10]. Schwarzian Derivative give some useful information about the behaviour on the transition points of parabolas, particularly at their critical points. Negativeness of Schwarzian Derivative is a sufficient condition for it to be chaotic. The Schwarzian derivative is given by,

$$Sf(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''x)}{f'(x)}\right)^2 \tag{12}$$

where $f'(x)$, $f''(x)$, $f'''(x)$ denote the first, second and third order derivatives of the $f(x)$ at x, respectively. If $Sf(x)<0$, $f(x)$ will have a chaotic behaviour on x and a values. Schwarzian Derivatives (13), (14) of functions, $f_1(x)$ and $f_2(x)$ as given (10) and (11) that are solutions of the differantial equation (9) are calculated bu using (12). 2-dimensional shapes are given in Fig. 2 and Fig. 3.

$$Sf_1(x)=-\frac{6a^2}{(2ax-a-1)^2} \tag{13}$$

$$Sf_2(x)=-\frac{6a^2}{(2ax-a+1)^2} \tag{14}$$

Schwarzian Derivatives $Sf_1(x)$ and $Sf_2(x)$ of $f_1(x)$, and $f_2(x)$ approach to highly chaos, between $x=0$ and $x=1$ in x-axis while a is getting greater.

3.2. Sensitive dependence on initial conditions

Lyapunov exponents of a dynamical system provide a quantitative measure of its sensitivity to initial conditions. The average rate of convergence or divergence of the system along the axes in phase space gives Lyapunov Spectrum of the map. We focus on the calculation of Lyapunov exponents in the current context of one-dimensional discrete maps. The Lyapunov exponent measures the exponential rate at which neighboring orbits are moving apart. It is determined by averaging the natural logarithm of the derivative evaluated along an orbit. If a dynamical system has sensitive dependence on initial conditions that is a typical x_0 is a sensitive point, then it cannot be used to predict for large time, because there are errors in numerical calculations. Hence this is an important concept for chaos. More precisely, let $f(x)$ be a map on R, a point x_0 has sensitive dependence on initial conditions, if there is a constant $d>0$, such that for any $\delta(n)>0$, there is an x satisfying $|x-x_0|<\delta(n)$ and an integer k, such that $|f^k(x)-f^k(x_0)|d$. Let f^k denotes the k th iterate of $f(x)$. For simplicity, we call such a point x_0 a sensitive point. If the initial condition is unstable, small errors or perturbations in the state would cause the orbit to move away from the fixed point.

We focus on the calculation of Lyapunov exponents in the current context of one-dimensional discrete maps [11-15]. Lyapunov exponents measure the rate of divergence of orbits originating from arbitrarily close initial conditions. That is, they measure a system's sensitivity to its initial conditions. A positive Lyapunov exponent indicates that the system is chaotic. $f_1(x)$ and $f_2(x)$, which are the solutions of the same non-linear dynamic system by given in (9) with initial condition x_0. Examine a small perturbation of this starting point, defined by $x_0+\delta_0$, where the initial separation δ_0 is assumed to be very small. Suppose δ_n is the separation after n iterations of the system. If $|\delta_n|\approx|\delta_0|e^{n\lambda}$, then λ is called a Lyapunov exponent. Lyapunov exponents can be found, for a trajectory starting at x_0, from the limit. The exponents are described as [15],

$$\lambda=\lim_{n\to\infty}[\frac{1}{n}\sum_{i=0}^{n-1}ln|f'(x_i)|] \tag{15}$$

n is the number of iteration of the dynamical system and x_0 is the initial condition. Further details can be found in [15]. It is clear from (15) that ì depends on the starting point x_0. In practice, the value of λ converges after a few hundreds iterations:

$$\lambda\approx\frac{1}{N}\sum_{i=0}^{N-1}ln|f'(x_i)| \tag{16}$$

$\lambda < 0$, the system attracts to a fixed point or stable periodic orbit. These systems are non conservative (dissipative) and exhibit asymptotic stability. $\lambda = 0$, the system is neutrally stable. Such systems are conservative and in a steady state mode. They exhibit Lyapunov stability. $\lambda > 0$, the system is chaotic and unstable. The exponents of $f_1(x)$ and $f_2(x)$ are given by (16),

$$\lambda_1(a) \approx \frac{1}{N}\sum_{i=0}^{N-1} ln\,|\,2ax_i - (a+1)\,| \tag{17}$$

$$\lambda_2(a) \approx \frac{1}{N}\sum_{i=0}^{N-1} ln\,|\,-2ax_i + (a-1)\,| \tag{18}$$

Fig. 4 shows the Lyapunov exponent computed for the map with a ranging from 0 to 2. For each value of a, (17) and (18) is estimated using $N = 1000$, with an initial starting point of $x_0 = 0.5$. This spectrum is invariant in a basin of attraction, and so will only vary in different regions of stability. In the current case, the signal is entirely chaotic, and then undergoes periodic transitions from chaos to stability, as a increases. Fig.4 shows the Lyapunov exponent computed for the $f_1(x)$ and $f_2(x)$, for $0 < a < 2$. We notice that λ remains negative for $a < 1.6$, and approaches 0 at the period doubling bifurcation.

A bifurcation diagram gives the value and stability of the steady state and periodic orbits. In bifurcation diagram, for each value of a is reported the local maximum of values of x_n. The transition from one regime to another is called a bifurcation [16]. A point in a bifurcation diagram where stability changes from stable to unstable is called a bifurcation point. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behaviour.A bifurcation is a sudden change in the number or nature of the fixed and periodic points of the system[17]. Fig.5 shows the stability of the solution as a function of a, and then its transition to unstable and chaotic behaviour. One way of summarizing the range of behaviours encountered when a increases is to construct a bifurcation diagram. Such a diagram gives the value and stability of the steady state and periodic orbits (Fig.5). In this diagram, for each value of a is reported the local maximum of values of x_n. The transition from one regime to another is called a bifurcation. Further system parameter changes are shown to result in even more extreme changes in behaviour, including higher periodicity, quasiperiodicity and chaos. Let $f_1(x) = f_1(x,a)$ and $f_2(x) = f_2(x,a)$, where a is a scalar parameter. The variable x is on the vertical axis, and the bifurcation parameter a is on the horizontal axis. As shown in Fig.5, transitions start between parabolas which is given (10) and (11) that are different solutions of one differential equation (9) with variable coefficients, when x goes to 0.5 and a is greater than 1.6.

4. Conclusion

In this paper, we present a approach for the characterization of the points on parabolic intersection seams as either local minimum or saddle points using same second order differential equation. The curvilinear coordinates are conceptually important, they also give rise to additional practical applications; electromagnetic coupling, vibration, turbulence, absorption, molecular motions. The parabolic intersections are not isolated points but rather are part of an extended seam of geometries where the energy of two states varies while preserving their degeneracy. Finally, the chaotic behaviour of strong interactions of parabolic intersections can be determined by using the methods of Schwarzian Derivative, Lyapunov Exponent, Bifuracation Diagram and these methods show good results.

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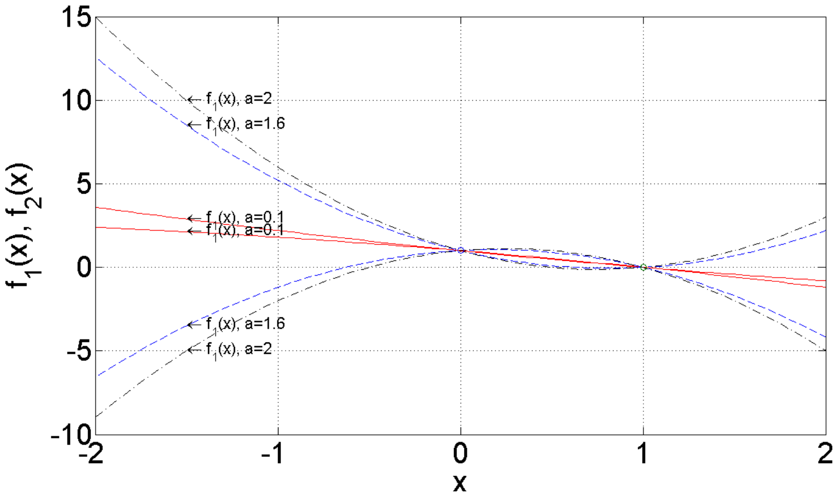

Figure 1: The Interactions Between Surfaces of Parabolic Intersections, $f_1(x), f_2(x)$, for various a

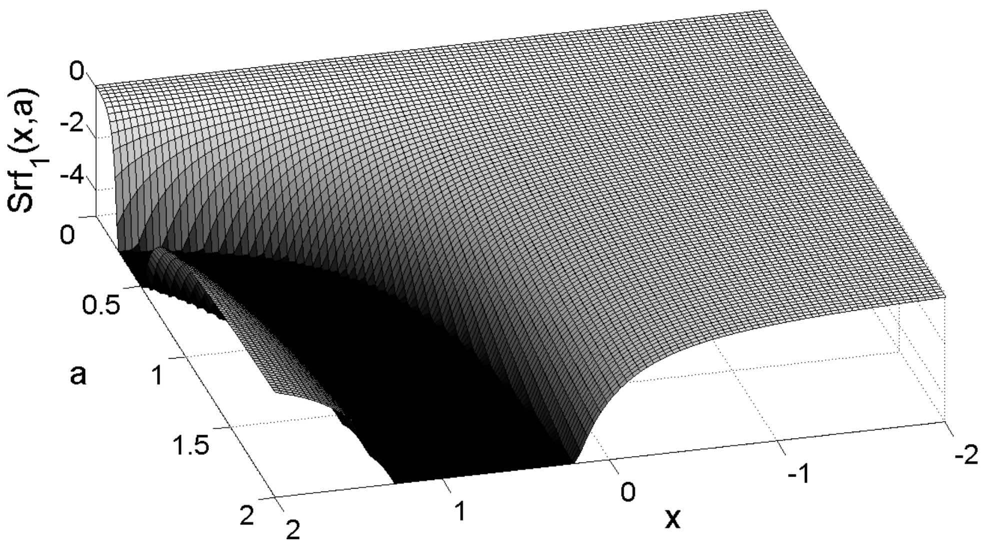

Figure 2: Schwarzian Derivative of $f_1(x)$ for various a

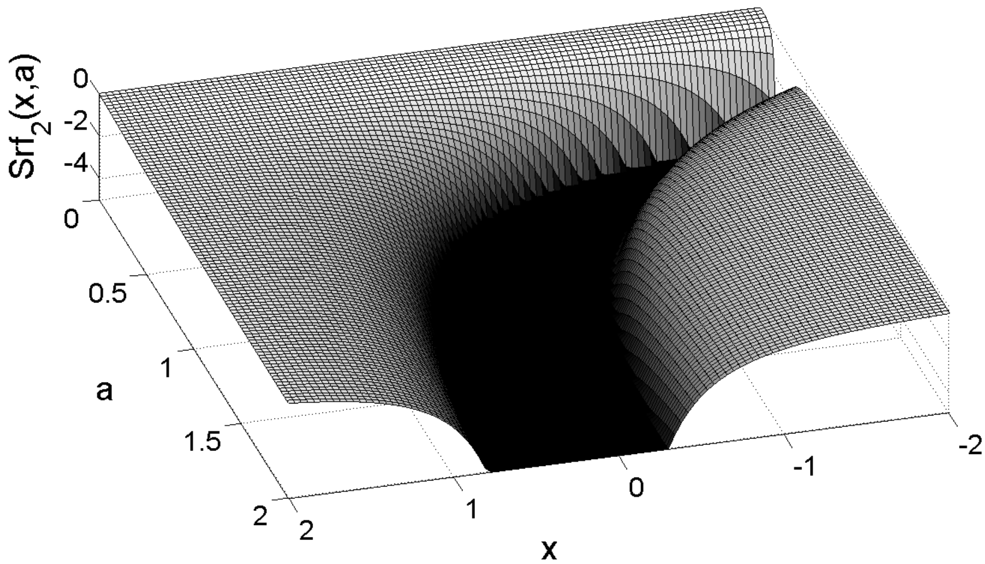

Figure 3: Schwarzian Derivative of $f_2(x)$ for various a

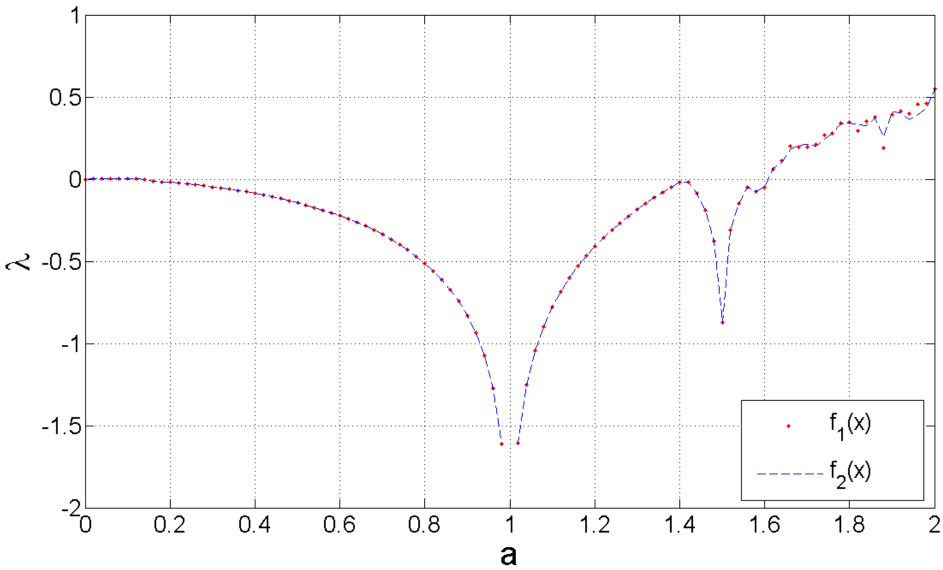

Figure 4: The Lyapunov Exponent for $0 < a < 2$, $f_1(x)$ *and* $f_2(x)$ for $x_0 = 0.5$

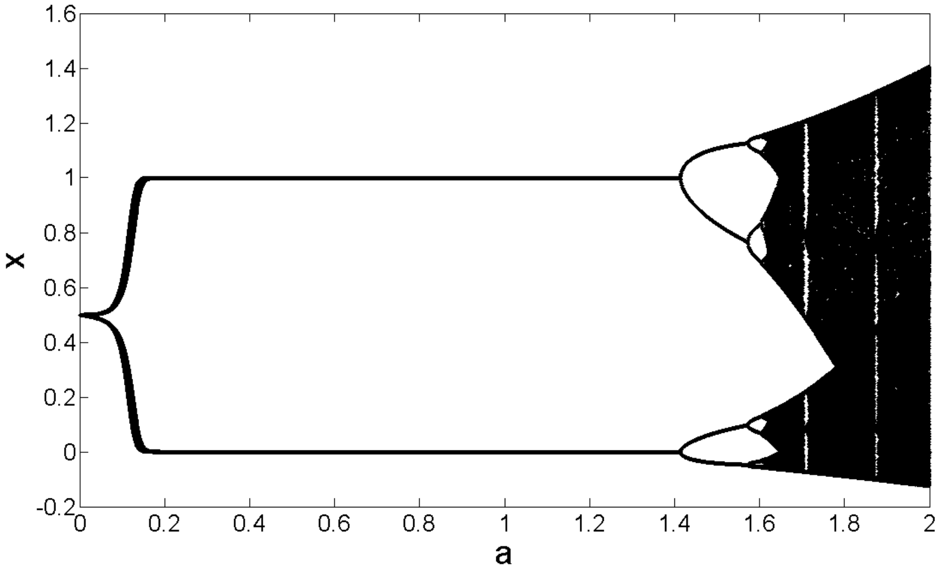

Figure 5: Bifurcation Diagram for $f_1(x)$ and $f_2(x)$ for $x_0 = 0.5$

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On a Special Type Nearly Quasi-Einstein Manifold

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Abstract: In the present paper, we consider a special type of nearly quasi-Einstein manifold denoted by $N(\mathrm{QE})_n$. Most of the sections are based on some properties of $N(\mathrm{QE})_n$. We give some theorems about these manifolds. In the last section, a special type nearly quasi-Einstein spacetime is investigated.

Keywords: Quasi-Einstein manifold, nearly quasi-Einstein manifold, spacetime.

1. Introduction

A non-flat n-dimensional Riemannian or a semi-Riemannian manifold (M,g) $(n>2)$ is said to be an Einstein manifold if the condition

$$S(X,Y)=\frac{r}{n}g(X,Y) \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M,g), respectively.

Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat n-dimensional Riemannian manifold (M,g) $(n>2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following condition

$$S(X,Y)=ag(X,Y)+bA(X)A(Y) \tag{1.2}$$

where $a,b\in\mathbb{R}$ and A is a non-zero 1-form such that

$$g(X,U)=A(X) \tag{1.3}$$

for all vector fields X on M, [4]. Then A is called the associated 1-form and U is called the generator of the manifold.

Also M.C. Chaki and R.K. Maity [1] studied the quasi-Einstein manifolds by considering a and b as scalars such that $b\neq 0$ and U as a unit vector field.

In 2008, U.C. De and A.K. Gazi [2] introduced the notion of nearly quasi-Einstein manifold. A non-flat n-dimensional Riemannian manifold (M,g) $(n>2)$ is called a nearly quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$S(X,Y) = ag(X,Y) + bE(X,Y) \tag{1.4}$$

where a and b are non-zero scalars and E is a non-zero symmetric tensor of type (0,2).

Then E is called the associated tensor and a and b are called the associated scalars of M. An n-dimensional nearly quasi-Einstein manifold is denoted by $N(\mathrm{QE})_n$. An example of $N(QE)_4$ has been given in [2].

The nearly quasi-Einstein manifolds have also studied by A.K. Gazi, U.C. De [5], D.G. Prakasha, C.S. Bagewadi [7] and R.N. Singh, M.K. Pandey, D. Gautam [8].

In [8], R.N. Singh, M.K. Pandey, D. Gautam consider a type of nearly quasi-Einstein manifold whose associated tensor E of type (0,2) is in the form

$$E(X,Y) = A(X)B(Y) + B(X)A(Y) \tag{1.5}$$

where A and B are non-zero 1-forms associated with orthogonal unit vector fields V and U, i.e.,

$$g(U,U) = 1, \quad g(V,V) = 1 \quad \text{and} \quad g(U,V) = 0. \tag{1.6}$$

These vector fields are defined by

$$g(X,U) = A(X), \quad g(X,V) = B(X)$$

for every vector field X.

In the present paper, we consider a special type of nearly quasi-Einstein manifold, $N(\mathrm{QE})_n$, whose associated tensor E is of the form (1.5) with the condition (1.6). Some theorems about this manifold are proved and some properties are obtained.

2. A Special Type Nearly Quasi-Einstein Manifold

In this section, we consider a special type of $N(\mathrm{QE})_n$ whose Ricci tensor satisfies the conditions (1.5) and (1.6), i.e., it satisfies the following condition

$$S(X,Y) = ag(X,Y) + b[A(X)B(Y) + B(X)A(Y)] \tag{2.1}$$

where A and B are non-zero 1-forms, a and b are non-zero scalars.

Definition 1. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the following condition

$$\nabla_X \xi = \rho X + \phi(X)\xi \tag{2.2}$$

where $X \in TM$, ϕ is a linear form and ρ is a function, [10].

In the local transcription, this reads

$$\nabla_i \xi^h = \rho\, \delta_i^h + \xi^h \phi_i \tag{2.3}$$

where ξ^h and ϕ_i are the components of ξ and ϕ, and δ_i^h is the Kronecker symbol.

A torse-forming vector field ξ is called

i) recurrent, if $\rho = 0$,

ii) concircular, if the form ϕ_i is a gradient covector, i.e., there is a function $\psi(x)$ such that $\phi = d\psi(x)$,

iii) convergent, if it is concircular and $\rho = const.exp(\psi)$.

Therefore, recurrent vector fields are characterized by the following equation

$$\nabla_X \xi = \phi(X)\xi. \tag{2.4}$$

Also, from the Definition 1, for a concircular vector field ξ, we get

$$(\nabla_Y \xi)X = \rho g(X,Y) \tag{2.5}$$

for all $X, Y \in TM$.

Theorem 2.1. Let V_n be a $N(QE)_n$ satisfying the condition (2.1) and let U and V be the vector fields corresponding to the associated 1-forms A and B, respectively. Thus, the vector fields U and V cannot be concircular vector fields.

Proof. We consider a special type $N(QE)_n$ satisfying the condition (2.1). Let U and V corresponding to the associated 1-forms A and B be concircular vector fields, respectively. In local coordinates, thus we have

$$\nabla_i A_j = \rho g_{ij} \tag{2.6}$$

and

$$\nabla_i B_j = \sigma g_{ij} \tag{2.7}$$

where ρ and σ are non-zero scalar functions.

Taking the covariant derivative of the condition $g(U,U) = 1$, it is found that

$$(\nabla_j A_i)A^i = 0 \tag{2.8}$$

where $A^i = g^{ih} A_h$ and h is the arbitrary choice for indexing and the summation runs from 1 to n.

Multiplying (2.6) by A^j and using the equation (2.8), we get

$$\rho A_i = 0$$

which contradicts to the fact that ρ is a non-zero scalar function and A is a non-zero 1-form. Similarly, it can be shown that the generator V cannot be a concircular vector field. In this case, $N(QE)_n$ satisfying the condition (2.1) does not admit concircular vector fields U and V corresponding to the associated 1-forms A and B, respectively. Hence, the proof is completed.

Definition 2. A quadratic conformal Killing tensor is defined as a second order symmetric tensor T satisfying the condition

$$\begin{aligned}(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) \quad &+ (\nabla_Z T)(X,Y) = \alpha(X)g(Y,Z) \\ &+ \alpha(Y)g(Z,X) + \alpha(Z)g(X,Y)\end{aligned} \tag{2.9}$$

where α is a 1-form, [9].

Now, we consider a $N(QE)_n$ admitting a generator vector as a torse-forming vector field and the other be not. If we assume that the generator U is a torse-forming vector field, then we have from (1.6) and (2.3)

$$\nabla_j A_i = \rho(g_{ij} - A_i A_j) \tag{2.10}$$

where ρ is a scalar function.

Taking the covariant derivative of the condition $g(U,V)=0$ and using the equation (2.10), it can be seen that

$$A^i(\nabla_k B_i) = -\rho B_k. \tag{2.11}$$

By the aid of (2.9), (2.10) and (2.11), we prove the following theorem.

Theorem 2.2. Let V_n be a $N(QE)_n$ satisfying the condition (2.1) and admitting the Ricci tensor as a quadratic conformal Killing tensor. If the vector field U generated by the 1-form A is a torse-forming vector field and the other vector field V generated by the 1-form B is not, then the vector field V is divergence-free.

Proof. Suppose that the Ricci tensor of a $N(QE)_n$ satisfying the condition (2.1) is a quadratic conformal Killing tensor. In this case, in local coordinates, we have from (2.9)

$$\nabla_k S_{ij} + \nabla_i S_{jk} + \nabla_j S_{ki} = \alpha_k g_{ij} + \alpha_i g_{jk} + \alpha_j g_{ki} \tag{2.12}$$

where α is a 1-form.

Taking the covariant derivative of (2.1), we get

$$\begin{aligned} \nabla_k S_{ij} &= a_k g_{ij} + b_k(A_i B_j + A_j B_i) \\ &+ b((\nabla_k A_i)B_j + A_i(\nabla_k B_j) + (\nabla_k A_j)B_i + A_j(\nabla_k B_i)) \end{aligned} \tag{2.13}$$

where a and b are the associated scalars of this manifold and $a_k = \partial_k a$, $b_k = \partial_k b$.

If the vector field U generated by the 1-form A is a torse-forming vector field, then we have the relation (2.10). Changing the indices by cyclic in (2.13), using (2.10) and (2.12), it can be obtained that

$$\begin{aligned} &(a_k + 2b\rho B_k - \alpha_k)g_{ij} + (a_i + 2b\rho B_i - \alpha_i)g_{jk} + (a_j + 2b\rho B_j - \alpha_j)g_{ik} \\ &+ b_k(A_i B_j + A_j B_i) + b_i(A_j B_k + A_k B_j) + b_j(A_k B_i + A_i B_k) \\ &+ b(A_i(\nabla_k B_j) + A_j(\nabla_k B_i) + A_j(\nabla_i B_k) + A_k(\nabla_j B_i) + A_k(\nabla_i B_j) + A_i(\nabla_j B_k)) \\ &- 2b\rho(A_i A_k B_j + A_j A_k B_i + A_i A_j B_k) = 0. \end{aligned} \tag{2.14}$$

Multiplying (2.14) by g^{ij} and considering (2.11), we get

$$\begin{aligned} &(n+2)(a_k + 2b\rho B_k - \alpha_k) + 2b_i(A^i B_k + A_k B^i) \\ &- 4b\rho B_k + 2b(A^i(\nabla_i B_k) + A_k(\nabla_i B^i)) = 0. \end{aligned} \tag{2.15}$$

Moreover, multiplying (2.15) by A^k and B^k, respectively, and using the condition (1.6), we obtain the following equations

$$(n+2)(a_k - \alpha_k)A^k + 2b_k B^k + 2b\nabla_k B^k = 0 \tag{2.16}$$

$$(n+2)(a_k - \alpha_k)B^k + 2nb\rho + 2b_k A^k = 0. \tag{2.17}$$

On the other hand, multiplying (2.14) by $A^i A^j A^k$ and using (2.11), it is found that

$$(a_k - \alpha_k)A^k = 0. \tag{2.18}$$

Multiplying (2.14) by $B^i B^j A^k$, we find

$$(a_k - \alpha_k)A^k + 2b_k B^k = 0. \tag{2.19}$$

Since b is a non-zero scalar function, from (2.16), (2.18) and (2.19), it can be seen that

$$\nabla_k B^k = 0.$$

Thus, the vector field V generated by the 1-form B is divergence-free. This completes the proof.

Definition 3. A non-flat n-dimensional Riemannian manifold (M, g) $(n > 2)$ is called a generalized Ricci-recurrent manifold if its Ricci tensor S of type (0,2) satisfies the condition

$$(\nabla_X S)(Y, Z) = \gamma(X)S(Y, Z) + \delta(X)g(Y, Z) \tag{2.20}$$

where γ and δ are non-zero 1-forms, [3]. If $\delta = 0$, then the manifold reduces to a Ricci-recurrent manifold, [6].

Theorem 2.3. Let $N(QE)_n$ be a generalized Ricci-recurrent manifold. Thus, the vector fields U and V generated by the 1-forms A and B cannot be torse-forming vector fields.

Proof. We consider that V_n is a $N(QE)_n$ satisfying the condition (2.1). In this case, in local coordinates, we have the equation (2.13) by Theorem 2.2. Let the vector field U generated by the 1-form A be a torse-forming vector field and the other be not. Then the relation (2.10) is satisfied. If we suppose that V_n is a generalized Ricci-recurrent manifold, by the aid of (2.10), (2.13) and (2.20), we obtain

$$\begin{aligned}(a_k - \delta_k - a\gamma_k)g_{ij} + (b_k - b\gamma_k)(A_i B_j + A_j B_i) + b[\rho(g_{ik} - A_i A_k)B_j \\ + A_i(\nabla_k B_j) + \rho(g_{jk} - A_j A_k)B_i + A_j(\nabla_k B_i)] = 0\end{aligned} \tag{2.21}$$

where γ_k and δ_k denote the components of the 1-forms γ and δ.

Multiplying (2.21) by g^{ij} and using the condition (2.11), it can be seen that

$$a_k = \delta_k + a\gamma_k. \tag{2.22}$$

Moreover, multiplying (2.21) by $A^i A^j$ and using (1.6), we get

$$a_k - \delta_k - a\gamma_k + 2bA^i(\nabla_k B_i) = 0. \tag{2.23}$$

By the aid of (2.11), (2.22) and (2.23), it is found that

$$b\rho B_k = 0$$

which contradicts to the fact that b and ρ are non-zero scalar functions and B is a non-zero 1-form. Therefore, the vector field U of this manifold cannot be a torse-forming vector field. By similar calculations it can be easily obtained that the vector field V of this manifold also cannot be a torse-forming vector field. Thus, the proof is completed.

3. A Special Type $N(QE)_n$ Spacetime

In this section, we will examine $N(QE)_4$ spacetime which will be denoted by $N(QES)_4$ satisfying the condition (2.1).

The Einstein field equations (EFE) without cosmological constant is written as the following form

$$kT(X,Y) = S(X,Y) - \frac{r}{2} g(X,Y) \tag{3.1}$$

where S is the Ricci tensor, r is the scalar curvature, g is the metric tensor, k is a constant and T is the energy-momentum tensor.

Theorem 3.1. In a $N(QES)_4$ satisfying the condition (2.1), the trace of the energy-momentum tensor is constant if and only if the associated scalar a is constant.

Proof. Let us consider a $N(QES)_4$ satisfying the condition (2.1). From (3.1) and (2.1), it is obtained that

$$kT(X,Y) = (a - \frac{r}{2})g(X,Y) + b(A(X)B(Y) + A(Y)B(X)). \tag{3.2}$$

Moreover, using (2.1), the scalar curvature of a $N(QES)_4$ is found as

$$r = 4a. \tag{3.3}$$

From (3.2) and (3.3), we have

$$kT(X,Y) = -ag(X,Y) + b(A(X)B(Y) + A(Y)B(X)). \tag{3.4}$$

Contracting (3.4) over X and Y, we obtain

$$\tilde{T} = -\frac{4}{k} a \tag{3.5}$$

where $\tilde{T}$ denotes the trace of the energy-momentum tensor.

It follows from (3.5) that if the associated scalar a is constant, then the trace of the energy-momentum tensor is constant. The converse is also true. Hence, the proof is completed.

Theorem 3.2. In a perfect fluid $N(QES)_4$ spacetime satisfying the condition (2.1) with the constant associated scalar a, the change of the isotropic pressure is proportional to the change of the energy density.

Proof. In a perfect fluid spacetime, the energy-momentum tensor is in the form

$$T(X,Y) = (\sigma + p)\lambda(X)\lambda(Y) + pg(X,Y) \tag{3.6}$$

where σ is the energy density, p is the isotropic pressure and λ is a non-zero 1-form such that $g(X,V) = \lambda(X)$ for all X, V being the velocity vector field of the flow, that is, $g(V,V) = -1$. Also, $\sigma + p \neq 0$.

Using (3.6) in (3.1) and contracting the resulting equation over X and Y, and considering the condition $g(V,V) = -1$ and (3.3), it can be seen that

$$3p - \sigma = -\frac{4}{k} a \tag{3.7}$$

where a is the associated scalar of the manifold and k is a constant.

If the associated scalar a of $N(QES)_4$ is constant, then taking the covariant derivative of the equation (3.7) yields

$$3\nabla_Z p = \nabla_Z \sigma \tag{3.8}$$

for all vector fields Z .

Thus, the change of the isotropic pressure is proportional to the change of the energy density. This completes the proof.

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The Numerical Solution of Fractional Differential-Algebraic Equations (FDAEs)

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Abstract: In this paper, numerical solution of Fractional Differential–Algebraic Equations (FDAEs) is studied. Firstly Fractional Differential–Algebraic Equations (FDAEs) have been converted to power series and then numerical solution of Fractional Differential–Algebraic Equations (FDAEs) is obtained.

Keywords: Differential-Algebraic Equations (DAEs), Fractional Differential-Algebraic Equation (FDAEs), Power Series.

1. Introduction

Fractional differential equations have gained importance and popularity during the past three decades because of its powerful potential applications. The applications of ordinary fractional differential equations or fractional differential-algebraic equations (FDAE) used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, chemical physics, electrochemistry, optics, polymer physics and signal processing can be successfully modelled by linear or nonlinear fractional differential equations. Meanwhile, some rich fractional dynamical motion which reflect the inherent nature of realistic physical systems are observed. In short, fractional calculus and fractional differential equations have played more and more important role in almost all the scientific fields. [1,4,5,8,12,13]

In this paper, the method is applied to solve FDAEs of the form with the initial conditions [11]

$$D_*^{\alpha_i} x_i(t) = f_i\left(t, x_1, x_2, \ldots, x_n, x_1', x_2', \ldots, x_n'\right), \quad i = 1,2,3,\ldots,n-1, \quad t \geq 0, \quad 0 < \alpha_i \leq 1$$

$$g(t, x_1, x_2, \ldots, x_n) = 0 \tag{1.1}$$

$$x_i(0) = a_i, \qquad i = 1,2,3,\ldots,n$$

2. Basic definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ [6], for example. Riemann-Liouville, Grunwald-Letnikow, Caputo and the generalized functions approach. The most commonly used definitions are those of Riemann-Liouville and Caputo. We give some basic definitions and properties of fractional calculus theory used in this paper.

Definition 2.1. A real function $f(x), x < 0.$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^P f_I(x)$, where $f_I(x) \in C[0,\infty)$. Clearly, $C_\mu \subset C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x), x < 0.$ is said to be in the space $C_\mu^m, m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function, $f \in C_\mu, \mu \geq -1$, is defined as [4].

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \alpha > 0, x > 0 \tag{2.1}$$

$$J^0 f(x) = f(x) \tag{2.2}$$

The properties of the operator f^α can be found in [6, 7]: we mention only the following.

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) \tag{2.3}$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x) \tag{2.4}$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \tag{2.5}$$

The Riemann- Liouville derivative has certain disadvantages when trying to model real-word phenomena using fractional differential equations. Therefore, we will introduce a modified fractional differential operator D_*^αr proposed by Caputo's work on the theory of viscoelasticity [10].

Definition 2.4. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{2.6}$$

for $m-1 < \alpha \leq m, \quad m \in N, \quad x > 0, \quad f \in C_{-1}^m$.

Also, we give two basic properties of its in here. [4].

Lemma 2.1. If $m-1 < m, m \in N$ and $f \in C_\mu^m, m \geq -1$, then

$$D_*^\alpha J^\alpha f(x) = f(x) \tag{2.7}$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{2.8}$$

3. Our Method

Consider the differential-algebraic equations (DAEs)

$$F(t, x, x') = 0 \tag{3.1}$$

with the initial condition

$$x(t_0) = x_0$$

where F and x are vector functions. The solutions of (3.1) can be assumed that

$$x = x_0 + et \tag{3.2}$$

where e is a vector function. Substitute (3.2) into (3.1) and neglect bigger order term. We have the linear equation of e in the form

$$Ae = B \tag{3.3}$$

where A and B are constant matrices. Solving this (3.3), the coefficients of e in (3.2) can be found. Repeating the above procedure for bigger terms, we can obtain the arbitrary order power series of the solutions for (3.1) [1,2,3,9].

4. Power series of solution for DAEs

We determine another type of power series in the form

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots + (f_n + p_1 e_1 + \cdots + p_m e_m)t^n \tag{4.1}$$

where $p_1, p_2, \cdots, p_m$ are constants. $e_1, e_2, \cdots, e_m$ are bases of vector e, m is the size of vector e. x is a vector with m elements in (3.2). Every element can be written by the power series in (4.1).

$$x_i = x_{i,0} + x_{i,1} t + x_{i,2} t^2 + \cdots + e_i t^n \tag{4.2}$$

where x_i is the i th element of x. Substituting (4.2) into (3.1), we can get the following expression:

$$f_i = (f_{i,n} + p_{i,1} e_1 + \cdots + p_{i,m} e_m)t^{n-j} + O(t^{n-j+1}) \tag{4.3}$$

where f_i is the i th element of $f(t, x, x')$ in (3.1) and j is 0 if $f(t, x, x')$ have x', 1 if do not. From (4.3) and (3.3), we can determine the linear equation in (3.3) as follows:

$$A_{i,j} = P_{i,j} \tag{4.4a}$$

$$B_i = -f_{i,n} \tag{4.4b}$$

solving this linear equation, we have e_i $(i = 1, \cdots, m)$. Substituting e_i into (4.2), we have x_i $(i = 1, \cdots, m)$ polynomials of degree n. Repeating this procedure from (4.4), we can get the arbitrary order power series of the solution for FDAEs in (1.1). If we repeat the above procedure, we have numerical solution of FDAEs in (1.1).

5. Numerical Examples

To express the effectiveness of the method, we consider the following fractional differential-algebraic equations. All the results were calculated by using the Maple software.

Example 5.1. We consider the following fractional differential-algebraic equation.

$$\begin{aligned} &D_*^{\alpha} x(t) - t y'(t) + x(t) - (1+t) y(t) = 0, \qquad 0 < \alpha \le 1, \\ &y(t) - \sin t = 0 \end{aligned} \tag{5.1}$$

with initial conditions $x(0) = 1$, $y(0) = 0$ and exact solutions $x(t) = e^{-t} + t \sin t$, $y(t) = \sin t$ when $\alpha = 1$.

From initial condition, the solutions of (5.1) can be supposed as

$$\begin{aligned} x(t) &= x_0 + e_1 t \quad \rightarrow \quad x(t) = 1 + e_1 t \\ y(t) &= y_0 + e_2 t \quad \rightarrow \quad y(t) = e_2 t \end{aligned} \tag{5.2}$$

Substituting (5.2) into (5.1) and neglecting higher order terms, we have

$$\begin{aligned} 1 + e_1 + O(t) &= 0 \\ (-1 + e_2)t + O(t^2) &= 0 \end{aligned} \tag{5.3}$$

These formulae correspond to (4.3). The linear equation that corresponds (4.4) can be given in the following:

$$Ae = B, \tag{5.4}$$

Where;

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

From Eq. (5.4) we have linear equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solving this linear equation, we have

$$e = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$\begin{aligned} x(t) &= 1 - t \\ y(t) &= t \end{aligned} \tag{5.5}$$

from (5.5) the solutions of (5.1) can be supposed as

$$\begin{aligned} x(t) &= 1 - t + e_1 t^2 \\ y(t) &= t + e_2 t^2 \end{aligned} \tag{5.6}$$

In like manner substituting (5.6) into (5.1) and neglecting higher order terms, we have

$$\begin{aligned} (-3 + 2e_1)t + O(t^2) &= 0 \\ -e_2 t^2 + O(t^3) &= 0 \end{aligned} \tag{5.7}$$

where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

From Eq. (5.7) we have linear equation

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

By solving this linear equation, we have

$$e = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$$

Therefore

$$x(t) = 1 - t + 3/2t^2$$
$$y(t) = t$$

Repeating the above procedure, we have

$$x^*(t) = 1 - t + 1.500000000\,t^2 - 0.1666666667\,t^3 - 0.1250000000\,t^4 - 0.0083333333333\,t^5$$
$$+ 0.009722222222\,t^6 - 0.00001984126984\,t^7$$
$$- 0.0001736111111\,t^8 - 0.2755731922\ \ 10^{-5}\ t^9$$

$$y^*(t) = t - 0.1666666667\,t^3 + 0.0083333333333\,t^5 - 0.00001984126984\,t^7 + 0.2755731922\ \ 10^{-5}\ t^9$$

Table 1. Numerical results of the solution in Example 5.1

	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	
t	$x^*(t)$	$x^*(t)$	$x^*(t)$	$x_{exact}(t)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.76429238	0.84929941	0.91482085	0.91482076
0.2	0.75450959	0.80166956	0.85846473	0.85846462
0.3	0.79031612	0.79789989	0.82947437	0.82947428
0.4	0.85249500	0.82508727	0.82608746	0.82608739
0.5	0.93232467	0.87601449	0.84624350	0.84624343
0.6	1.02420517	0.94545816	0.88759718	0.88759712
0.7	1.12379061	1.02907565	0.94753775	0.94753768
0.8	1.22732913	1.12295936	1.02321382	1.02321384
0.9	1.33139163	1.22343656	1.11156381	1.11156388
1.0	1.43275528	1.32697596	1.20935035	1.20935043

Table 1 shows the approximate solutions for Eq. (5.1) obtained for different values of α using our method. The results are in good agreement with the results of the exact solutions.

Example 5.2: Consider the following fractional differential-algebraic equation.

$$\begin{aligned} & x(t) + y(t) = e^{-t} + \sin t \\ & D_*^{\alpha} x(t) + x(t) - y(t) = -\sin t, \qquad 0 < \alpha \le 1, \end{aligned} \tag{5.8}$$

with initial conditions $x(0) = 1$, $y(0) = 0$ and exact solutions in this case $x(t) = e^{-t}$, $y(t) = \sin t$ when $\alpha = 1$.

Repeating the above procedure, we have obtained the numerical results shown in Table 2 by using Maple 15 software.

Table 2. Numerical results of the solution in Example 5.2

	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	
t	$x^*(t)$	$x^*(t)$	$x^*(t)$	$x_{exact}(t)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.76089099	0.83739311	0.90483738	0.90483741
0.2	0.69092614	0.74943903	0.81873062	0.81873075
0.3	0.63965013	0.68161285	0.74081815	0.74081822
0.4	0.59708770	0.62503221	0.67031998	0.67032004
0.5	0.55999258	0.57601215	0.60653064	0.60653065
0.6	0.52688938	0.53262381	0.54881712	0.54881163
0.7	0.49696401	0.49371280	0.49658769	0.49658530
0.8	0.46970221	0.45851976	0.44932904	0.44932896
0.9	0.44474480	0.42650762	0.40656968	0.40656965
1.0	0.42182078	0.39727365	0.36787945	0.36787944

Table 2 shows the approximate solutions for Eq. (5.2) obtained for different values of α using our method. The results are in good agreement with the results of the exact solutions.

6. Conclusion

In this study, the present method has been extended to solve fractional differential-algebraic equations (FDAEs). Two examples are given to demonstrate to powerfulness of the method. The results obtained by the method are in good-agreement with the exact solutions. The study shows that the method is a reliable technique to solve fractional differential–algebraic equations, and offer notable advantages from the points of applicability, computational costs, and accuracy.

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A Solution Proposal to Indefinite Quadratic Interval Transportation Problem

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Abstract: The data of real world applications generally cannot be expressed strictly. An efficient way of handling this situation is expressing the data as intervals. Thus, this paper focus on the Indefinite Quadratic Interval Transportation Problem (IQITP) in which all the parameters i.e. cost and risk coefficients of the objective function, supply and demand quantities are expressed as intervals. A Taylor series approach is presented for the solution of IQITP by means of the expression of intervals with its left and right limits. Also a numerical example is executed to illustrate the procedure.

Keywords: Quadratic Interval Transportation Problem, Interval coefficients, Taylor Series.

1. Introduction

Transportation Problem (TP) has wide practical applications in logistic systems, manpower planning, personnel allocation, inventory control, production planning, etc. and aims to find the best way to fulfill the demand of n demand points using the capacities of m supply points. The parameters of the transportation problem are unit costs (or profits), supply and demand quantities. The unit cost is the coefficient of the objective function and it could represent transportation cost, average delivery time of the commodities, number of goods transported unfulfilled demand, product deterioration, preference coefficient, and many others. The linear functions are the most useful and widely used in operational research. Also quadratic functions and quadratic problems are the least difficult ones to handle out of all nonlinear programming problems. A fair number of functional relationships occurring in the real world are truly quadratic. For example kinetic energy carried by a rocket or an atomic particle is proportional to the square of its velocity, in statistics, the variance of a given sample of observations is a quadratic function of the values that constitute the sample. So there are countless other non-linear relationships occurring in nature, capable of being approximated by quadratic functions.

Indefinite quadratic programming problems and Interval Transportation Problem have been extensively studied for several decades. A bibliography of Quadratic programming problems can be found in [11]. Using fuzzy triangular technique, [1] proposed a fuzzy method to solve interval transportation problems. Interval Fractional Transportation Problem (IFTP) in which all the parameters i.e. cost and preference coefficients of the objective function, supply and demand quantities are expressed as intervals. A Taylor series approach is presented for IFTP by means of the expression of intervals with its left and right limits in [2]. Sivri et al. [3] proposed a Taylor series based method to IFTP whose objective function coefficients are assumed as intervals. Also in [4], a new approach is proposed by the variable transformation for a linear fractional programming problem with interval coefficients in the objective function. In [5], a fuzzy multi objective linear fractional programming problem is reduced to a single objective problem using the Taylor series and an approximate solution is obtained. In [6,9,10,12], the authors studied fixed charge indefinite quadratic transportation problems and fixed charge bi-criterion quadratic transportation problems. Guzel and Sivri [8] concerned with the multi objective version of transportation problem, and proposed a solution procedure based on Taylor series expansion. In [13], using fuzzy technique, a new method is proposed for interval transportation problems by considering the right bound and midpoint of interval. Also in [14], fuzzy and interval programming technique is presented to deal with inexact coefficient in multi objective programming problem.

This paper dealt with the IQITP in which all the parameters are expressed as intervals. Expressing the parameters as interval makes Decision Maker (DM) more comfortable and this enables to consider tolerances for the model parameters in a more natural and direct way. Therefore, IQITP seems to be more realistic and reliable according to crisp values. In this paper, we present an iterative procedure based on the Taylor series expansion. Firstly, a feasible initial point is determined within the Northwest Corner method by means of expressing all the interval parameters as left and right limits. Then the objective function is linearized by using first order Taylor series expansion about the feasible initial point. Thus IQITP is transformed a traditional linear programming problem. And then an iterative procedure is presented in such a way that the optimal solution of lastly constructed linear programming problem is selected as the point where about the objective will be expanded to its first order Taylor series in the next iteration step. The stopping criterion of the proposed procedure is obtaining the same point for the last two iteration steps. A numerical example supports the proposed procedure.

2. Indefinite Quadratic Interval Transportation Problem

The mathematical formulation of IQITP can be stated as follows:

$$\min Z(\mathbf{x}) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left[c_{ij}^L, c_{ij}^R \right] x_{ij} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left[d_{ij}^L, d_{ij}^R \right] x_{ij} \right) \quad \text{(P1)}$$

s.t. $$\sum_{j=1}^{n} x_{ij} = \left[a_i^L, a_i^R \right] \quad i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = \left[b_j^L, b_j^R \right] \quad j = 1, 2, \ldots, n,$$

$$x_{ij} \geq 0 \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.$$

x_{ij} is decision variable which refers to product quantity that transported from supply point *i* to demand point *j*. The closed interval $\left[c_{ij}^L, c_{ij}^R \right]$ denotes that the unit transportation cost from *i* th supply point to *j* th demand point lies between c_{ij}^L and c_{ij}^R. The closed interval $\left[d_{ij}^L, d_{ij}^R \right]$ denotes that the depreciation (risk) by transport from *i* th supply point to *j* th demand point lies between d_{ij}^L and d_{ij}^R. The closed interval $\left[a_i^L, a_i^R \right]$ represent that *i* th supply quantity lies between a_i^L and a_i^R. Similarly, the closed interval $\left[b_j^L, b_j^R \right]$ represent that *j* th demand quantity lies between b_j^L and b_j^R

In the above problem the cost of transporting one unit from *i* th origin to *j* th destination is $\sum_{i=1}^{m} \sum_{j=1}^{n} \left[c_{ij}^L, c_{ij}^R \right] x_{ij}$, but while transporting goods can get damaged so the total cost of damaged good is $\sum_{i=1}^{m} \sum_{j=1}^{n} \left[d_{ij}^L, d_{ij}^R \right] x_{ij}$. Our aim is to minimize the two cost simultaneously: therefore we consider the product of two cost.

We note that the objective function being the product of two affine function is a quasi concave function will have its optimal solution at an extreme point.

Correspondingly to the literature, the model in this paper has the following assumptions:

- $\sum_{i=1}^{m} a_i^L = \sum_{j=1}^{n} b_j^L$ and $\sum_{i=1}^{m} a_i^R = \sum_{j=1}^{n} b_j^R$ (Balance condition)
- The parameters $a_i^L, b_j^L, c_{ij}^L, d_{ij}^L, a_i^R, b_j^R, c_{ij}^R, d_{ij}^R$ are all nonnegative.

3. A Taylor Series Approach for IQITP

To apply the Taylor series approach, we need to specify an initial single point from the feasible region of (P1). First interval supply-demand quantities and interval quadratic objective function coefficients are converted into deterministic ones by means of the combination of their's left and right limit in the following way:

$$\left[a_i^L, a_i^R \right] \Rightarrow \bar{a}_i = \delta_i a_i^R + (1-\delta_i) a_i^L = a_i^L + \left(a_i^R - a_i^L \right) \delta_i \quad (i = 1, 2, \ldots, m). \quad (1)$$

$$\left[b_j^L, b_j^R \right] \Rightarrow \bar{b}_j = \mu_j b_j^R + (1-\mu_j) b_j^L = b_j^L + \left(b_j^R - b_j^L \right) \mu_j \quad (j = 1, 2, \ldots, n). \quad (2)$$

$$\left[c_{ij}^L, c_{ij}^R \right] \Rightarrow \bar{c}_{ij} = \theta_{ij} c_{ij}^R + (1-\theta_{ij}) c_{ij}^L = c_{ij}^L + \left(c_{ij}^R - c_{ij}^L \right) \theta_{ij} \quad (i = 1, 2, \ldots, m)(j = 1, 2, \ldots, n) \quad (3)$$

$$\left[d_{ij}^L, d_{ij}^R \right] \Rightarrow \bar{d}_{ij} = \lambda_{ij} d_{ij}^R + (1-\lambda_{ij}) d_{ij}^L = d_{ij}^L + \left(d_{ij}^R - d_{ij}^L \right) \lambda_{ij} \quad (i = 1, 2, \ldots, m)(j = 1, 2, \ldots, n) \quad (4)$$

where $\delta_i, \mu_j, \theta_{ij}, \lambda_{ij} \in [0,1]$. With these equivalent expression of the interval parameters, (P1) is converted to the following IQITP:

$$\begin{aligned} \min Z &= Z_1(x).Z_2(x) \\ &= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^L + \theta_{ij}(c_{ij}^R - c_{ij}^L) x_{ij} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^L + \lambda_{ij}(d_{ij}^R - d_{ij}^L) x_{ij} \right) \quad \text{(P2)} \\ &= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \bar{c}_{ij} x_{ij} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \bar{d}_{ij} x_{ij} \right) \end{aligned}$$

s.t. $$\sum_{j=1}^{n} x_{ij} = a_i^L + \left(a_i^R - a_i^L \right) \delta_i \quad i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j^L + \left(b_j^R - b_j^L\right)\mu_j \qquad j = 1,2,\dots,n,$$

$$x_{ij} \ge 0\,,\ \delta_i, \mu_j, \theta_{ij}, \lambda_{ij} \in [0,1]. \quad i = 1,2,\dots,m, \quad j = 1,2,\dots,n.$$

The main purpose here is to specify an initial feasible point, not an optimal one. Thus, the value of the combination parameters $\delta_i, \mu_j, \theta_{ij}$ and λ_{ij} ($i = 1,2,\dots,m,\quad j = 1,2,\dots,n$) can be chosen arbitrarily from the interval $[0,1]$. After substituting these arbitrary values in (P2), a traditional TP is obtained and then an initial basic feasible solution can be determined by Northwest Corner Method which ignores the objective function coefficients and compute a basic feasible solution of TP, where the basic variables are selected from the North – West corner (i.e. top left corner). Let denote the initial feasible solution as $\mathbf{X}^{(0)} = \left(\mathbf{x}^{(0)}, \boldsymbol{\theta}^{(0)}, \boldsymbol{\lambda}^{(0)}, \boldsymbol{\delta}^{(0)}, \boldsymbol{\mu}^{(0)}\right)$.

Using the first order Taylor series at the feasible point $\mathbf{X}^{(0)}$, the objective function of (P2) can be constructed approximately as follows:

$$Z \approx Z\left(\mathbf{X}^{(0)}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \left.\frac{\partial Z}{\partial x_{ij}}\right|_{\mathbf{X}^{(0)}} \left(x_{ij} - x_{ij}^{(0)}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \left.\frac{\partial Z}{\partial \theta_{ij}}\right|_{\mathbf{X}^{(0)}} \left(\theta_{ij} - \theta_{ij}^{(0)}\right) + \sum_{i=1}^{m}\sum_{j=1}^{n} \left.\frac{\partial Z}{\partial \lambda_{ij}}\right|_{\mathbf{X}^{(0)}} \left(\lambda_{ij} - \lambda_{ij}^{(0)}\right)$$

Hence the terms $Z\left(\mathbf{X}^{(0)}\right)$, $\sum_{i=1}^{m}\sum_{j=1}^{n} \left.\frac{\partial Z}{\partial x_{ij}}\right|_{X^{(0)}} \left(x_{ij}^{(0)}\right)$, $\sum_{i=1}^{m}\sum_{j=1}^{n} \left.\frac{\partial Z}{\partial \theta_{ij}}\right|_{X^{(0)}} \left(\theta_{ij}^{(0)}\right)$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} \left.\frac{\partial Z}{\partial \lambda_{ij}}\right|_{X^{(0)}} \left(\lambda_{ij}^{(0)}\right)$ are constant value, all of these do not change the direction of minimization and can be eliminated. The first partial derivatives with respect to the variables x_{ij}, θ_{ij}, λ_{ij} in the Taylor series expansion are:

$$\frac{\partial Z}{\partial x_{ij}} = \frac{\partial Z_1}{\partial x_{ij}} Z_2 + \frac{\partial Z_2}{\partial x_{ij}} Z_1 = \bar{c}_{ij} Z_2 + \bar{d}_{ij} Z_1\,,$$

$$\frac{\partial Z}{\partial \theta_{ij}} = \frac{\partial Z_1}{\partial \theta_{ij}} Z_2 + \frac{\partial Z_2}{\partial \theta_{ij}} Z_1 = \left(c_{ij}^R - c_{ij}^L\right) x_{ij} Z_2\,,$$

$$\frac{\partial Z}{\partial \lambda_{ij}} = \frac{\partial Z_1}{\partial \lambda_{ij}} Z_2 + \frac{\partial Z_2}{\partial \lambda_{ij}} Z_1 = \left(d_{ij}^R - d_{ij}^L\right) x_{ij}\ Z_1\,.$$

Thus, an equivalent form of (P2) can be constructed as follows:

$$\begin{aligned}\min \bar{Z} \approx & \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\bar{c}_{ij} Z_2 + \bar{d}_{ij} Z_1\right)\Big|_{\mathbf{X}^{(0)}}\ x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\left(c_{ij}^R - c_{ij}^L\right) x_{ij} Z_2\right)\Big|_{\mathbf{X}^{(0)}}\ \theta_{ij} \\ & + \sum_{i=1}^{m}\sum_{j=1}^{n} \left(\left(d_{ij}^R - d_{ij}^L\right) x_{ij}\ Z_1\right)\Big|_{\mathbf{X}^{(0)}}\ \lambda_{ij}\end{aligned} \qquad \left(\mathrm{P3}\big|_{\mathbf{X}^{(0)}}\right)$$

$$\text{s.t.} \qquad \sum_{j=1}^{n} x_{ij} = a_i^L + \left(a_i^R - a_i^L\right)\delta_i \quad i = 1,2,\dots,m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j^L + \left(b_j^R - b_j^L\right)\mu_j \qquad j = 1,2,\dots,n,$$

$$x_{ij} \ge 0\ ,\ \delta_i, \mu_j, \theta_{ij}, \lambda_{ij} \in [0,1],\ \ i = 1,2,\dots,m, \quad j = 1,2,\dots,n,$$

$$\sum_{i=1}^{m} a_i \ge \sum_{j=1}^{n} b_j\ .$$

We note here that since the objective function does not depend on the variables δ_i and μ_j, The partial derivatives with respect to these variables are equal to zero and so it is not necessary to add these to the objective function.

The last constraint of equation $\left(\mathrm{P3}\big|_{\mathbf{X}^{(0)}}\right)$ guarantees that total demand is certainly met. Thus, IQITP is converted to a linear programming problem $\left(\mathrm{P3}\big|_{\mathbf{X}^{(0)}}\right)$ which can be easily solve with any computer packages. Let denote the optimal solution of equation $\left(\mathrm{P3}\big|_{\mathbf{X}^{(0)}}\right)$ as $\mathbf{X}^{(1)}$. If the objective function of (P2) is expanded to its first Taylor polynomial at the new point $\mathbf{X}^{(1)}$, the problem $\left(\mathrm{P3}\big|_{\mathbf{X}^{(1)}}\right)$ can be constructed, similarly. Let denote the optimal solution of $\left(\mathrm{P3}\big|_{\mathbf{X}^{(1)}}\right)$ by $\mathbf{X}^{(2)}$. The objective value at the point $\mathbf{X}^{(2)}$ is better than the value at $\mathbf{X}^{(1)}$. Hence the last obtained point is a closer extreme point to the optimal solution of (P2), this procedure can be continued until the last point is repeated. So the optimal solution of (P2) is obtained by repeating the given procedure.

The Taylor series approach can be summarized with the following algorithm:

Step 0: (Initialization) After constructing (P2), obtain a feasible initial point $\mathbf{X}^{(0)}$ with the Northwest Corner Method for any value of $\delta_i, \mu_j \in [0,1]$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Set $k = 0$.

Step 1: (Generating a new point) With the aim of linearizing the indefinite quadratic objective, use first order Taylor Series Expansion about the point $\mathbf{X}^{(k)}$, build and solve the corresponding $\left(\mathrm{P3}\big|_{\mathbf{X}^{(k)}}\right)$ and obtained its optimal solution set $\mathbf{X}^{(k+1)}$.

Step 2: (Stopping criterion) If $\mathbf{X}^{(k)} = \mathbf{X}^{(k+1)}$ then stop. Otherwise, set $k = k+1$ and go to Step 1.

4. A numerical example

Let us consider the following interval objective functions:

$$Z_1(\mathbf{x}) = [1,2]x_{11} + [2,4]x_{12} + [1,3]x_{13} + [3,5]x_{14} + [0,2]x_{21} + [2,5]x_{22} + [1,4]x_{23} + [3,4]x_{24} + [0,3]x_{31} + [1,2]x_{32} + [3,5]x_{33} + [2,4]x_{34}$$

$$Z_2(\mathbf{x}) = [2,3]x_{11} + [1,2]x_{12} + [3,5]x_{13} + [1,3]x_{14} + [0,4]x_{21} + [1,5]x_{22} + [2,3]x_{23} + [3,5]x_{24} + [0,5]x_{31} + [0,1]x_{32} + [1,3]x_{33} + [2,4]x_{34}$$

$$\sum_{j=1}^{4} x_{1j} = [18,24], \quad \sum_{j=1}^{4} x_{2j} = [10,17], \quad \sum_{j=1}^{4} x_{3j} = [20,26],$$

$$\sum_{i=1}^{3} x_{i1} = [10,19], \quad \sum_{i=1}^{3} x_{i2} = [7,12], \quad \sum_{i=1}^{3} x_{i3} = [16,20], \quad \sum_{i=1}^{3} x_{i4} = [15,19]$$

$$x_{ij} \geq 0 \quad i = 1,2,3 \quad j = 1,2,3,4.$$

After expressing all interval parameters in the form of (1)-(4), corresponding problem (P2) is constructed as follows:

$$Z_1(x) = (1+\theta_{11})x_{11} + (2+2\theta_{12})x_{12} + (1+2\theta_{13})x_{13} + (3+2\theta_{14})x_{14} + (2\theta_{21})x_{21} + (2+3\theta_{22})x_{22} + (1+3\theta_{23})x_{23} + (3+\theta_{24})x_{24} + (3\theta_{31})x_{31} + (1+\theta_{32})x_{32} + (3+2\theta_{33})x_{33} + (2+2\theta_{34})x_{34}$$

$$Z_2(x) = (2+\lambda_{11})x_{11} + (1+\lambda_{12})x_{12} + (3+2\lambda_{13})x_{13} + (1+2\lambda_{14})x_{14} + (4\lambda_{21})x_{21} + (1+4\lambda_{22})x_{22} + (2+\lambda_{23})x_{23} + (3+2\lambda_{24})x_{24} + (5\lambda_{31})x_{31} + (\lambda_{32})x_{32} + (1+2\lambda_{33})x_{33} + (2+2\lambda_{34})x_{34}$$

$$\min Z(\mathbf{x}) = \begin{pmatrix} (1+\theta_{11})x_{11} + (2+2\theta_{12})x_{12} + (1+2\theta_{13})x_{13} + (3+2\theta_{14})x_{14} \\ +(2\theta_{21})x_{21} + (2+3\theta_{22})x_{22} + (1+3\theta_{23})x_{23} + (3+\theta_{24})x_{24} \\ +(3\theta_{31})x_{31} + (1+\theta_{32})x_{32} + (3+2\theta_{33})x_{33} + (2+2\theta_{34})x_{34} \end{pmatrix} \cdot \begin{pmatrix} (2+\lambda_{11})x_{11} + (1+\lambda_{12})x_{12} + (3+2\lambda_{13})x_{13} + (1+2\lambda_{14})x_{14} \\ +(4\lambda_{21})x_{21} + (1+4\lambda_{22})x_{22} + (2+\lambda_{23})x_{23} + (3+2\lambda_{24})x_{24} \\ +(5\lambda_{31})x_{31} + (\lambda_{32})x_{32} + (1+2\lambda_{33})x_{33} + (2+2\lambda_{34})x_{34} \end{pmatrix} \text{ s.t.}$$

$$\sum_{j=1}^{4} x_{1j} = 18 + 6\delta_1, \quad \sum_{j=1}^{4} x_{2j} = 10 + 7\delta_2, \quad \sum_{j=1}^{4} x_{3j} = 20 + 6\delta_3$$

$$\sum_{i=1}^{3} x_{i1} = 10 + 9\mu_1, \quad \sum_{i=1}^{3} x_{i2} = 7 + 5\mu_2, \quad \sum_{i=1}^{3} x_{i3} = 16 + 4\mu_3, \quad \sum_{i=1}^{3} x_{i4} = 15 + 4\mu_4$$

$$x_{ij} \geq 0, \; \delta_i, \mu_j, \theta_{ij}, \lambda_{ij} \in [0,1]. \quad i = 1,2, \quad j = 1,2,3$$

Assuming the arbitrary value of δ_i, μ_j as one for $\forall i, \forall j$, the following initial feasible solution set $\mathbf{X}^{(0)}$ is determined by Northwest Corner Method:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 10 & 7 & 1 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 5 & 15 \end{bmatrix}, \boldsymbol{\theta}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{\lambda}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{\delta}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \boldsymbol{\mu}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

For the point $\mathbf{X}^{(0)}$, the values of objective is calculated as $Z_1\big|_{\mathbf{X}^{(0)}} = 80$, $Z_2\big|_{\mathbf{X}^{(0)}} = 85$ and so $Z\big|_{\mathbf{X}^{(0)}} = \left(Z_1\big|_{\mathbf{X}^{(0)}}\right)\left(Z_2\big|_{\mathbf{X}^{(0)}}\right) = 6800$. The corresponding problem $\left(\mathrm{P3}\big|_{\mathbf{X}^{(0)}}\right)$ can be written as follows:

$$
\begin{aligned}
\min \bar{Z} \approx & \left(\left(1+\theta_{11}\right) Z_2+\left(2+\lambda_{11}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{11}+\left(\left(2+2\theta_{12}\right) Z_2+\left(1+\lambda_{12}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{12} \\
& +\left(\left(1+2\theta_{13}\right) Z_2+\left(3+2\lambda_{13}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{13}+\left(\left(3+2\theta_{14}\right) Z_2+\left(1+2\lambda_{14}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{14} \\
& +\left(\left(2\theta_{21}\right) Z_2+\left(4\lambda_{21}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{21}+\left(\left(2+3\theta_{22}\right) Z_2+\left(1+4\lambda_{22}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{22} \\
& +\left(\left(1+3\theta_{23}\right) Z_2+\left(2+\lambda_{23}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{23}+\left(\left(3+\theta_{24}\right) Z_2+\left(3+2\lambda_{24}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{24} \\
& +\left(\left(3\theta_{31}\right) Z_2+\left(5\lambda_{31}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{31}+\left(\left(1+\theta_{32}\right) Z_2+\left(\lambda_{32}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{32} \qquad \text{s.t.} \\
& +\left(\left(3+2\theta_{33}\right) Z_2+\left(1+2\lambda_{33}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{33}+\left(\left(2+2\theta_{34}\right) Z_2+\left(2+2\lambda_{34}\right) Z_1\right)\Big|_{\mathbf{X}^{(0)}} x_{34} \\
& +\left(x_{11} Z_2\right)\Big|_{\mathbf{X}^{(0)}} \theta_{11}+\left(2 x_{12} Z_2\right)\Big|_{\mathbf{X}^{(0)}} \theta_{12}+\left(2 x_{13} Z_2\right)\Big|_{\mathbf{X}^{(0)}} \theta_{13}+\left(3 x_{23} Z_2\right)\Big|_{\mathbf{X}^{(0)}} \theta_{23} \\
& +\left(2 x_{33} Z_2\right)\Big|_{\mathbf{X}^{(0)}} \theta_{33}+\left(2 x_{34} Z_2\right)\Big|_{\mathbf{X}^{(0)}} \theta_{34}+\left(x_{11} Z_1\right)\Big|_{\mathbf{X}^{(0)}} \lambda_{11}+\left(x_{12} Z_1\right)\Big|_{\mathbf{X}^{(0)}} \lambda_{12} \\
& +\left(2 x_{13} Z_1\right)\Big|_{\mathbf{X}^{(0)}} \lambda_{31}+\left(x_{23} Z_1\right)\Big|_{\mathbf{X}^{(0)}} \lambda_{23}+\left(2 x_{33} Z_1\right)\Big|_{\mathbf{X}^{(0)}} \lambda_{33}+\left(2 x_{34} Z_1\right)\Big|_{\mathbf{X}^{(0)}} \lambda_{34}
\end{aligned}
$$

$$\sum_{j=1}^{4} x_{1j}=18+6\delta_1 \ , \quad \sum_{j=1}^{4} x_{2j}=10+7\delta_2 \ , \quad \sum_{j=1}^{4} x_{3j}=20+6\delta_3$$

$$\sum_{i=1}^{3} x_{i1}=10+9\mu_1 \ , \quad \sum_{i=1}^{3} x_{i2}=7+5\mu_2 \ , \quad \sum_{i=1}^{3} x_{i3}=16+4\mu_3 \ , \quad \sum_{i=1}^{3} x_{i4}=15+4\mu_4$$

$$x_{ij} \geq 0\,, \ \delta_i, \mu_j, \theta_{ij}, \lambda_{ij} \in [0,1]. \quad i=1,2, \quad j=1,2,3\,,$$

$$(18+6\delta_1)+(10+7\delta_2)+(20+6\delta_3) \geq (10+9\mu_1)+(7+5\mu_2)+(16+4\mu_3)+(15+4\mu_3).$$

The optimal solution set $\mathbf{X}^{(1)}$ of the problem $\left(\mathrm{P3}\big|_{\mathbf{X}^{(0)}}\right)$ is:

$$\mathbf{x}^{(1)}=\begin{bmatrix} 0 & 0 & 3 & 15 \\ 0 & 0 & 13 & 10 \\ 13 & 7 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\theta}^{(1)}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\lambda}^{(1)}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{\delta}^{(1)}=\begin{bmatrix} 0 & 0.4286 & 0 \end{bmatrix},$$

$$\boldsymbol{\mu}^{(1)}=\begin{bmatrix} 0.3333 & 0 & 0 & 0 \end{bmatrix}.$$

For the point $\mathbf{X}^{(1)}$, the values of objective is calculated as $Z_1\big|_{\mathbf{X}^{(1)}}=68$, $Z_2\big|_{\mathbf{X}^{(1)}}=50$ and so $Z\big|_{\mathbf{X}^{(1)}}=\left(Z_1\big|_{\mathbf{X}^{(1)}}\right)\left(Z_2\big|_{\mathbf{X}^{(1)}}\right)=3400$. The next optimal solution set $\mathbf{X}^{(2)}$ is the same with the previous solution set $\mathbf{X}^{(1)}$, i.e . therefore $\mathbf{X}^{(2)}=\mathbf{X}^{(1)}$. Thus algorithm ends.The last solution implies following interval values for the two objective functions:

$$Z_1\big|_{\mathbf{X}^{(2)}}=[68,80]$$
$$Z_2\big|_{\mathbf{X}^{(2)}}=[50,85].$$

5. Conclusion

In this paper, we deal with IQITP whose objective coefficients and supply-demand quantities are given as intervals. In real life applications, this version of indefinite quadratic transportation problem is more realistic and reliable according to crisp ones. For the proposed solution procedure, all the interval parameters are handled by means of combination of left and right limits. And after determining a feasible initial point with the Northwest Corner method, then the indefinite quadratic objective is linearized by using first order Taylor series expansion about the feasible initial point. Thus IQITP is transformed a traditional linear programming problem. And then an iterative procedure is executed in such a way that the optimal solution of lastly constructed linear programming problem is selected as the point where about the nonlinear objective will be expanded to its first order Taylor series in the next iteration step. The stopping criterion of the proposed procedure is obtaining the same point for the last two iteration step. Finally, a numerical example is provided to illustrate the proposed procedure.

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The Jacobsthal Sequences in The Groups Q_{2^n}, $Q_{2^n} \times_\varphi \mathbb{Z}_{2m}$ and $Q_{2^n} \times \mathbb{Z}_{2m}$

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Abstract: In [8], Deveci et.al defined the generalized order-k Jacobsthal orbit $J_A^k(G)$ of a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \cdots, a_k\}$ to be the sequence $\{x_i\}$ of the elements of G such that

$$x_i = a_{i+1} \text{ for } 0 \le i \le k-1, \quad x_{i+k} = \begin{cases} (x_i)^2 (x_{i+1}), & k = 2, \\ (x_i)\cdots(x_{i+k-2})^2 (x_{i+k-1}), & k \ge 3 \end{cases} \quad \text{for } i \ge 0.$$

The length of the period of the generalized order-k Jacobsthal orbit $J_A^k(G)$ is denoted by $LJ_A^k(G)$ and is called the generalized order-k Jacobsthal length of G [8].

In this study, we obtain the generalized order-k Jacobsthal lengths of the quarternion group Q_{2^n}, the semidirect product $Q_{2^n} \times_\varphi \mathbb{Z}_{2m}$ and the direct product $Q_{2^n} \times \mathbb{Z}_{2m}$ for $m, n \ge 3$.

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1 Introduction and Preliminaries

The well-known Jacobsthal sequence $\{J_n\}$ is defined by the following recurrence relation:

for $n \ge 2$

$$J_n = J_{n-1} + 2J_{n-2} \tag{1.1}$$

where $J_0 = 0$ and $J_1 = 1$.

In [13], Koken and Bozkurt showed that the Jacobsthal numbers are also generated by a matrix

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}.$$

Kalman [11] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \cdots, c_{k-1}$ are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In [15], Yilmaz and Bozkurt defined the k sequences of the generalized order-*k* Jacobsthal numbers as follows:

for $n > 0$ and $1 \leq \mathrm{i} \leq k$

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + \ldots + J_{n-k}^i, \tag{1.2}$$

with initial conditions

$$J_n^i = \begin{cases} 1 & \text{if } n = 1\text{-}i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \leq n \leq 0,$$

where J_n^i is the nth term of the ith sequence. If $k=2$ and $i=1$ the generalized order-*k* Jacobsthal sequence is reduced to the conventional Jacobsthal sequence.

In [15], Yilmaz and Bozkurt showed that

$$\begin{bmatrix} J_{n+1}^i \\ J_n^i \\ J_{n-1}^i \\ \vdots \\ J_{n-k+2}^i \end{bmatrix} = C \cdot \begin{bmatrix} J_n^i \\ J_{n-1}^i \\ J_{n-2}^i \\ \vdots \\ J_{n-k+1}^i \end{bmatrix} \tag{1.3}$$

where C is called the generalized order-*k* Jacobsthal matrix and C is a *k*-square matrix as following:

$$C = \begin{bmatrix} 1 & 2 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.4}$$

Also, it was obtained that $B_n = C \cdot B_{n-1}$ where

$$B_n == \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^k \\ \vdots & \vdots & & \vdots \\ J_{n-k+1}^1 & J_{n-k+1}^2 & \cdots & J_{n-k+1}^k \end{bmatrix}. \tag{1.5}$$

Lemma 1.1 (Yilmaz and Bozkurt [15]). Let C and B_n be as (1.4) and (1.5), respectively. Then, for all integers $n \geq 0$

$$B_n = C^n.$$

Reducing the generalized order-k Jacobsthal sequence $(k \geq 2)$ by a modulus m, we can get the repeating sequences, denoted by

$$\{J_n^{k,m}\} = \{J_{1-k}^{k,m}, J_{2-k}^{k,m}, \cdots, J_0^{k,m}, J_1^{k,m}, \cdots, J_i^{k,m}, \cdots\}$$

where $J_i^{k,m} \equiv J_i^k \ (\mathrm{mod}\ m)$. It has the same recurrence relation as in (1.2) [8].

Theorem 1.1 (Deveci et al [8]). The sequence $\{J_n^{k,m}\}$ $(k \geq 2)$ is periodic.

The notation $hJ^{k,m}$ denotes the smallest period of $\{J_n^{k,m}\}$ $(k \geq 2)$ [8].

Theorem 1.2 (Deveci et.al [8]). If p is a prime such that $p \neq 2$, then $hJ^{k,p^a} = \left|\langle C \rangle_{p^a}\right|$.

The usual notation $G_1 \times_\varphi G_2$ is used for the semidirect product of the group $G_1$by G_2, where $\varphi : G_2 \to \mathrm{Aut}(G_1)$ is a homomorphism such that $b\varphi = \varphi_b$ and $\varphi_b : G_1 \to G_1$ is an element of $\mathrm{Aut}(G_1)$.

The quaternion group Q_{2^n}, $(n \geq 3)$ are defined by presentation

$$Q_{2^n} = \left\langle x, y : x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \right\rangle.$$

Let $m, n \geq 3$ be integers. By the definitions of the direct and semidirect products, we get the following presentations:

$$Q_{2^n} \times \mathbb{Z}_{2m} = \left\langle x, y, z : x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xyx = z^{2m} = [x,z] = [y,z] = e \right\rangle,$$

$$Q_{2^n} \times_\varphi \mathbb{Z}_{2m} = \left\langle x, y, z : x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xyx = z^{2m} = e, z^{-1}xzx = e, z^{-1}yzy = e \right\rangle,$$

where if $\mathbb{Z}_{2m} = \langle z \rangle$, then $\varphi : \mathbb{Z}_{2m} \to \mathrm{Aut}(Q_{2^n})$ is a homomorphism such that $z\varphi = \varphi_z$; $\varphi_z : Q_{2^n} \to Q_{2^n}$ is defined by $x\varphi_z = x$ and $y\varphi_z = y^{-1}$

For more information see [9,10].

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,e,f,a,b,c,d,e,f,a,b,c,d,e,f,\cdots$ is simply periodic with period 6.

Many references may be given for some special linear recurrence sequences in groups and related issues; see for example, [1-7,9,12,14,16]. Deveci et.al [8] expanded the theory to the Jacobsthal sequence. In this study, we obtain the generalized order-k Jacobsthal lengths of the quarternion group Q_{2^n}, the semidirect product $Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}$ and the direct product $Q_{2^n} \times \mathbb{Z}_{2m}$ $(m, n \geq 3)$ for initial (seeds) sets y, x and y, x, z.

2 Main Results and Proofs

Definition 2.1. Let $hJ^{k,m}_{(a_1,a_2,\cdots a_k)}$ denote the smallest period of the integer-valued recurrence relation $u_n = u_{n-1} + 2u_{n-2} + \cdots + u_{n-k}$, $u_1 = a_1, u_2 = a_2, \cdots, u_k = a_k$ when each entry is reduced modulo m.

Theorem 2.1. Let $a_1, a_2, \cdots, a_k, x_1, x_2, \cdots, x_k \in \mathbb{Z}$ and let p be a prime with $p \neq 2$, $\gcd(a_1, a_2, \cdots, a_k, p) = 1$ and $\gcd(x_1, x_2, \cdots, x_k, p) = 1$. Then we have

$$hJ^{k,p}_{(a_1,a_2,\cdots a_k)} = hJ^{k,p}_{(x_1,x_2,\cdots x_k)}.$$

Proof. Let $hJ^{k,p} = \left|\langle C\rangle_p\right| = r$. From (1.3), we can write $\begin{bmatrix} u_{n+r} \\ u_{n+r-1} \\ \vdots \\ u_{n+r-k+1} \end{bmatrix} = C^r \cdot \begin{bmatrix} u_n \\ u_{n+r-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix}$. So, we get $\begin{bmatrix} u_{n+r} \\ u_{n+r-1} \\ \vdots \\ u_{n+r-k+1} \end{bmatrix} \equiv \begin{bmatrix} u_n \\ u_{n+-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix} \bmod p$, in the natural way. Thus the proof is completes.

Theorem 2.2. $LJ^2_{(y,x)}\left(Q_{2^n}\right) = hJ^{2,2^{n-1}}$.

Proof. The orbit $J^2_{(y,x)}\left(Q_{2^n}\right)$ is

$$y, x, x^{2^{n-2}+1}, \cdots.$$

It is clear from Theorem 2.1 that this sequence has period $hJ^{2,2^{n-1}}$.

Theorem 2.3. $LJ^3_{(y,x,z)}\left(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}\right) = \operatorname{lcm}\left(2^{n-2}-7, hJ^{3,2m}\right)$.

Proof. The orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}\right)$ is

$$y, x, z, yx^2z, yxz^3, x^{-1}y^{-1}z^6, x^{2^{n-2}-1}z^{13}, yx^2z^{28}, xz^{60}, x^{-2}z^{129},$$
$$yx^2z^{277}, yxz^{595}, x^{-3}y^{-1}z^{1278}, x^{-2^{n-2}+1}z^{2745}, yz^{5896}, x^{2^{n-1}-3}z^{12664}, \cdots.$$

Using the above information, the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}\right)$ becomes:

$$x_0 = y, x_1 = x, x_2 = z, \cdots,$$
$$x_{13} = x^{-2^{n-2}+1}z^{2745}, \ x_{14} = yz^{5896}, x_{15} = x^{2^{n-1}-3}z^{12664}, x_{15} = z^{27201}, \cdots$$
$$x_{14i-1} = x^{-2^{n-2}+1}z^{J^3_{14,i-3}}, x_{14i} = z^{J^3_{14i-2}}y, x_{14i+1} = x^{2^{n-1}-4i+1}z^{J^3_{14i-1}}, x_{14i+2} = z^{J^3_{14i}}, \cdots.$$

So we need an i such that $x_{14i} = y, x_{14i+1} = x, x_{14i+2} = z$. if we choose $i = 2^{n-3}$, then we obtain

$$x_{2^{n-2}\cdot 7} = z^{J^3_{2^{n-2}\cdot 7-2}} y,\ x_{2^{n-2}\cdot 7+1} = xz^{J^3_{2^{n-2}\cdot 7-1}},\ x_{2^{n-2}\cdot 7+2} = z^{J^3_{2^{n-2}\cdot 7}},\ \cdots,$$

where $J^3_{2^{n-2}\cdot 7-k+1}$ and $J^3_{2^{n-2}\cdot 7-k+2}$ are even integers and $J^3_{2^{n-2}\cdot 7-k+3}$ is an odd integer. So, the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}\right)$ can be said to form layers of length $2^{n-2}\cdot 7$. It is easy to see that the orbit has period $\operatorname{lcm}\left(2^{n-2}-7, hJ^{3,2m}\right)$.

Theorem 2.4. $LJ^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right) = \operatorname{lcm}\left(7, hJ^{3,2m}\right)$.

Proof. The orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right)$ is

$$\begin{aligned}&y,\ x,\ z,\ yx^2z,\ yxz^3,\ yx^{2^{n-2}+1}z^6,\ x^{2^{n-1}}z^{13},\ yz^{28},\ xz^{60},\ z^{129},\\&yx^2z^{277},\ yxz^{595},\ yx^{2^{n-2}+1}z^{1278},\ x^{2^{n-1}}z^{2745},\ yz^{5896},\ xz^{12664},\cdots.\end{aligned}$$

Using the above information, the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right)$ becomes:

$$\begin{aligned}&x_0 = yz^{J^3_{-1}},\ x_1 = xz^{J^3_0},\ x_2 = z^{J^3_1},\ \cdots,\\&x_7 = yz^{J^3_6},\ \ x_8 = yz^{J^3_7},\ x_9 = z^{J^3_8},\\&x_{14} = yz^{J^3_{13}},\ x_{15} = xz^{J^3_{14}},\ x_{15} = z^{J^3_{15}},\ \cdots\\&x_{7\cdot i} = yz^{J^3_{7\cdot i-1}},\ x_{7\cdot i+1} = xz^{J^3_{7\cdot i}}y,\ x_{7i+2} = z^{J^3_{7i+1}},\ \cdots.\end{aligned}$$

The sequence can be said to form layers of length 42. So we need an i such that $x_{7\cdot i} = y, x_{7\cdot i+1} = x, x_{7\cdot i+2} = z$. It is easy to see that the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right)$ has period $\operatorname{lcm}\left(7, hJ^{3,2m}\right)$.

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A Note on Essential Subsemimodules

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Abstract: Let M be an R-semimodule and N non-zero subsemimodule of M. We say that N is an essential subsemimodule of M, if $N \cap K \neq (0)$ for every nonzero subsemimodule K of M. In this paper we study some useful results on essential subsemimodules and singular semimodule of semiring.

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1 Introduction

The paper is concerned with generalizing some results in ring theory and module theory. In this paper we will discuss about an extension of the notion of an essential subsemimodules of semimodules. The semiring and semimodule are important structures that have achieved an importance in recent development of theory as their usefulness to many disciplines has been discovered and exploited. Subsemimodules in semimodules are different from submodules in modules in that there are several kinds of submodules. In this paper we study some useful results on essential subsemimodules and singular semimodule of semiring. There are many different definitions of a semiring appearing in the literature. For definitions and properties of semirings, ideals, the reader is referred to [2].

Definition 1.1: A semiring is a set R together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(R, +)$ is a commutative monoid with identity element 0_R; $(R, \cdot)$ is a monoid with identity element 1; multiplication distributes over addition from either side and 0 is multiplicative absorbing, that is, $a \cdot 0 = 0 \cdot a = 0$ for each $a \in R$ [2].

Definition 1.2: A semiring R is said to have a unity if there exists $1_R \in R$ such that $1_R \cdot a = a \cdot 1_R = a$ for each $a \in R$ [2].

Definition 1.3: An ideal I of a semiring R will be called subtractive (k-ideal) if for $a \in I, a + b \in I, b \in R$ imply $b \in I$ [2].

For e.g.: The set $\mathbb{N}$of non-negative integers with the usual operations of addition and multiplication of integers is a semiring with $1_{\mathbb{N}}$.

Definition 1.4: Let R be a semiring. A left R-semimodule is a commutative monoid $(M, +)$ with additive identity 0_M for which we have a function $R \times M \rightarrow M$ defined by $(r, m) \hookrightarrow r \cdot m$ and called scalar multiplication which satisfies the following conditions for all r and r' of R and all elements m and m' of M,

1. $(r \cdot r')m = r(r' \cdot m)$

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2. $r \cdot (m + m') = r \cdot m + r \cdot m'$
3. $(r + r') \cdot m = r \cdot m + r' \cdot m$
4. $1_R \cdot m = m$ (If exists)
5. $r \cdot 0_M = 0_M = 0_R \cdot m.$

Convention: In this paper all semirings considered will be assumed to be commutative semirings with unity [2].

1 Essential Ideal

Definition 2.1: An ideal I of a semiring R is said to be an essential ideal of R if $I \cap K \neq 0$ for every nonzero ideal $K$of R [1].

Notation: We shall denote an essential ideal I of a semiring R by $I \lhd\cdot R$.
Proposition 2.2: If $0 \neq K \lhd\cdot I \lhd\cdot R$ and $\overline{K}$ is the ideal of R generated by K, then $\overline{K}$ is essential ideal of R [3].
Proof: Let L be any nonzero ideal of R. Since I is essential ideal in R, we have $I \cap L \neq 0$. Since K is essential ideal in I, we must have $0 \neq K \cap (I \cap L) \subseteq K \cap L \subseteq \overline{K} \cap L$. Thus $\overline{K}$ is essential ideal in R.
2 Essential Subsemimodules

Definition 3.1: Let M be an R-semimodule and N a non-zero subsemimodule of M. We say that N is an essential subsemimodule of M, if $N \cap K \neq (0)$ for every nonzero subsemimodule K of M.

Notation We shall denote an essential subsemimodule N of an R-semimodule M by $N \subseteq_e M$.

Clearly, that is equivalent to say $N \cap Rx \neq (0)$ for any nonzero element $x \in M$. So in particular, a nonzero left ideal I of R is an essential left ideal of R if $I \cap J \neq (0)$ for any nonzero left ideal J of R, which is equivalent to the condition $I \cap Rx \neq (0)$ for any nonzero element $r \in R$.

Proposition 3.2: Let M be a left R-semimodule. Any subsemimodule of M which contains an essential subsemimodule of M is itself essential in M.

Proposition 3.3: Let M be a left R-semimodule. If K is an essential subsemimodules of L and L is an essential subsemimodule of M then K is essential in M.

Proposition 3.4: Let M be a left R-semimodule. Let a be a non-zero element of M and let K be an essential subsemimodules of M then there is essential left ideal L of R such that $aL \neq 0$ and $aL \subseteq K$.

Proposition 3.5: Let$M$an$R$-semimodule and suppose that $N_1, N_2, \cdots, N_k$ are subsemimodules of M.then $\bigcap_{i=1}^{k} N_i \subseteq_e M$ if and only if $N \subseteq_e M$ for all i.

Proof: We only need to prove the proposition for $k = 2$. If $N_1, N_2 \subseteq_e M$, then $N_1 \subseteq_e M$ and $N_2 \subseteq_e M$ because both N_1 and N_2 contain $N_1 \bigcap N_2$.

Conversely, let P be a nonzero subsemimodule of M. Then $N_1 \cap P \neq 0$ because $N_1 \subseteq_e M$ and therefore $(N_1 \cap N_2) \cap P = N_2 \cap (N_1 \cap P) \neq 0$ because $N_2 \subseteq_e M$. Hence the proof.

3 Main Result

Definition 4.1: Let M be an R-semimodule and $x \in M$. The left **annihilator** of x in R is defined by $ann(x) = \{ r \in R \mid rx = 0\}$. Which is obviously a left ideal of R.Now, consider the set $Z(M) = \{x \in M \mid ann(x) \subseteq_e R\}$. It is easy to see that $Z(M)$ is a subsemimodule of M and we will call it the **singular subsemimodule** of M. If $Z(M) = M$, then M is called singular. If $Z(M) = 0$, then M is called **nonsingular [2]**.

Proposition 4.2: If $M = K/L$ for some R-semimodule K and some subsemimodule$L \subseteq_e K$. Then an R-semimodule M is singular. Anticipation

Proof: Suppose first that $M = K/L$ where K is an R-semimodule and$L \subseteq_e K$. Let $x = a + L \in M$ and let J be a nonzero left ideal of R. If $Ja = (0)$, then $Ja \subseteq L$ and so $ann(x) \cap J = J \neq (0)$. If $Ja \neq (0)$, then $L \cap Ja \neq (0)$ because $L \subseteq_e K$. So there exists $r \in J$ such that $0 \neq ra \in L$. That means $0 \neq r \in ann(x) \cap J$. So we have proved that $x \in Z(M)$ and hence $Z(M) = M$ i.e. M is singular.

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A note on "Exact solutions for nonlinear integral equations by a modified homotopy perturbation method"

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Abstract: In the paper "Exact solutions for nonlinear integral equations by a modified homotopy perturbation method" by A. Ghorbani and J. Saberi-Nadjafi, Computers and Mathematics with Applications, 56, (2008) 1032-1039, the authors introduced a new modification of the homotopy perturbation method to solve nonlinear integral equations.We discuss here the restrictions on their method for solving nonlinear integral equations. We also prove analytically that the method given by Ghorbani and Saberi-Nadjafi is equivalent to the series solution method when selective functions are polynomials.

Keywords: Modified Homotopy perturbation method; nonlinear integral equations; series solution method.

1 Introduction

Recently in [1], Ghorbani and Saberi-Nadjafi proposed a new modification of the homotopy perturbation method for solving nonlinear integral equations.

In this note, we show by an example that this method is not true generally. The purpose of this paper to show that, the new modification of the homotopy perturbation method is applicable for special case of nonlinear integral equations when the exact solution must appear as part of given function in integral equation otherwise this method is equivalence of the series solution method. This paper is organized as follow: The principle of the new modification of the homotopy perturbation method is described in Section 2. Two examples are studied in Section 3. The general remarks are given in Section 4.

2 The Principle of the New Modification of the Homotopy Perturbation Method

In [1], Ghorbani and Saberi-Nadjafi consider the following type of nonlinear integral equations:

$$y(x) = g(x) + \int_a^x k(x,t)\,[y(t)]^r dt, \qquad a \le x,\ t \le b,\ r \ge 2 \tag{2.1}$$

$$y(x) = g(x) + \int_a^b k(x,t)\,[y(t)]^r dt, \qquad a \le x,\ t \le b \tag{2.2}$$

Based on the Homotopy perturbation method (HPM) [4,5], they presented a method which called, modified HPM by them. In this regards, they rewrite (2.1) as:

$$y(x) = \sum_{m=0}^{N} \alpha_m\, v_m(x) - \sum_{m=0}^{N} \alpha_m\, v_m(x) + g(x) + \int_a^x k(x,t)\,[y(t)]^r dt \tag{2.3}$$

where α_m and $v_m(x)$, $m = 0,1,2,\cdots,N$ are called by them as the accelerating components of the parameter and selective functions, respectively.

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Based on the HPM, by selecting $F(u) = u - \sum_{m=0}^{N} \alpha_m \, v_m(x)$ they defined the following convex homotopy:

$$H_\alpha(u,p) = u(x) - pg(x) - (p-1)\sum_{m=0}^{N} \alpha_m \, v_m(x) - p\int_a^x k(x,t)\,[y(t)]^r dt = 0 \tag{2.4}$$

where the embedding parameter $p \in (0,1]$ can be considered as an expanding parameter. The HPM uses the embedding parameter p as a " small parameter", and writes the solution of (2.4) as a power series of p, i.e.,

$$u = u_0 + u_1 p + u_2 p^2 + \cdots, \tag{2.5}$$

Setting $p = 1$ results in the approximate solution of (2.4):

$$y = lim_{p\to 1} u = u_0 + u_1 + u_2 + \cdots, \tag{2.6}$$

Substituting Eq. (2.5) into (2.4) and equating the terms with identical powers of p, we can obtain a series of equations of the following form:

$$\begin{aligned}
&p^0: \; u_0 - \sum_{m=0}^{N} \alpha_m \, v_m(x) = 0,\\
&p^1: \quad u_1 - g(x) - \sum_{m=0}^{N} \alpha_m \, v_m(x) - \int_a^x k(x,t)\,[\,u_0(t)]^r dt = 0,\\
&p^2: \quad u_2 - \int_a^x k(x,t)\, H(\,u_0, u_1) dt = 0,\\
&p^3: \quad u_3 - \int_a^x k(x,t)\, H(\,u_0, u_1, \qquad u_2) dt = 0,\\
&\vdots
\end{aligned} \tag{2.7}$$

where $H(\,u_0, u_1, \cdots, u_j)$ depend upon $u_0, u_1, \cdots, u_j$.The $H\big(\,u_0, u_1, \cdots, u_j\big)$, calculate using Adomian formula [2,6]

$$H\big(\,u_0, u_1, \cdots, u_j\big) = \frac{1}{j!}\frac{\partial^j}{\partial p^j}\Big(\sum_{i=0}^{j} u_i \, p^i\Big)^r \Big|_{p=0}. \tag{2.8}$$

Which is called first time by Ghorbani as He polynomials [3]. However this formula has been used before Ghorbani's definition by first author and others in HPM [6] as Adomian polynomials (For more detail see [7]). It is obvious that the system of nonlinear equations in (2.7) is easy to solve and the components $u_i, i \geq 0$ of the homotopy perturbation method can be completely determined and the series solutions are thus entirely determined.

Remark. We get $\alpha_m, m = 0,1,2,\cdots,N$, and $v_m(x), m = 0,1,2,\cdots,N$ where $v_m(x)$ is form of function $g(x)$ accordingly we will obtain the exact solution, if we could not find $\alpha_m, m = 0,1,2,\cdots,N$ with $v_m(x), m = 0,1,2,\cdots,N$ so this method is not effective and this is a weakness in [1]. But if we increased $N, N \longrightarrow \infty$, we will obtain exact solution by Taylor series method. We discussed an example (2.2) about this case.

3 Examples

Example 1: [1,8] Consider the following nonlinear Volterra integral equation

$$y(x) = 1 - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4 + \int_0^x y^3\,(t) dt, \tag{3.1}$$

with the exact solution $y(x) = 1 + x$.

We apply this new modified HPM. We get $v_0(x) = 1, v_1(x) = x$ then

$$H_\alpha(u,p) = u(x) - p\left(1 - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4\right) + (p-1)(\alpha_0 + \alpha_1 x) - p\int_0^x [y(t)]^3\, dt = 0 \tag{3.2}$$

In view of Eq. (2.7) we have

$$\begin{aligned}
&p^0: \quad u_0(x) - \alpha_0 - \alpha_1 x = 0 \ \rightarrow \ u_0(x) = \alpha_0 + \alpha_1 x, \\
&p^1: \quad u_1(x) - p\left(1 - \tfrac{3}{2}x^2 - x^3 - \tfrac{1}{4}x^4\right) + \alpha_0 + \alpha_1 x - \int_0^x [u_0(t)]^3\,dt = 0, \\
&\rightarrow \ u_1(x) = 1 - \alpha_0 + ({\alpha_0}^3 + \alpha_1)x + \left(\tfrac{3}{2}{\alpha_0}^2\alpha_1 - \tfrac{3}{2}\right)x^2 + (\alpha_0{\alpha_1}^2 - 1)x^3 + \left(\tfrac{1}{4}{\alpha_1}^3 - \tfrac{1}{4}\right)x^4, \\
&p^{n+1}: \quad u_{n+1}(x) - \int_0^x H_n(t)\,dt = 0 \quad \rightarrow \quad u_{n+1}(x) = \int_0^x H_n(t)\,dt \quad n \geq 1.
\end{aligned} \tag{3.3}$$

To find $\alpha_m,\ m = 0,1$ in such a way that $u_1 = 0$. If $u_1 = 0$ then $u_2 = u_3 = \cdots = 0$, and the exact solution will be obtained as $y(x) = u_0(x)$. hence for all values of x we have

$$\begin{cases}
1 - \alpha_0 = 0, \\
{\alpha_0}^3 + \alpha_1 = 0, \\
\dfrac{3}{2}{\alpha_0}^2\alpha_1 - \dfrac{3}{2} = 0, \\
\alpha_0{\alpha_1}^2 - 1 = 0, \\
\dfrac{1}{4}{\alpha_1}^3 - \dfrac{1}{4} = 0.
\end{cases}$$

Solving the above algebraic equations, we have $\alpha_0 = \alpha_1 = 1$. Therefore the solution will be

$$y(x) = u_0(x) = \alpha_0 + \alpha_1 x = 1 + x$$

which is the same as the exact solution.

Example 2: [9] Consider the following nonlinear Volterra integral equation

$$y(x) = x + \int_0^x y^2(t)\,dt = 0 \tag{3.3}$$

with the exact solution $y(x) = tan(x)$.

We apply this new modified HPM [1] and get $v_0 = 1$ and $v_1(x) = x$. Then

$$H_\alpha(u,p) = u(x) - px + (p-1)(\alpha_0 + \alpha_1 x) - p\int_0^x [y(t)]^2\,dt = 0, \tag{3.4}$$

In view of Eq. (2.7) we have

$$\begin{aligned}
&u_0(x) = \alpha_0 + \alpha_1 x, \\
&u_1(x) = -\alpha_0 - \alpha_1 x + x + \int_0^x (\alpha_0 + \alpha_1 x)^2\,dt = 0, \\
&u_{n+1}(x) = \int_0^x \sum_{j=0}^{n} u_j\,u_{n-j}\,dt \qquad n \geq 1.
\end{aligned} \tag{3.5}$$

Now we find $\alpha_m,\ m = 0,1$ in such a way that $u_1 = 0$. If $u_1 = 0$ then $u_2 = u_3 = \cdots = 0$, and the exact solution will be obtained as $y(x) = u_0(x)$. hence for all values of x we have

$$\begin{cases}
-\alpha_0 = 0, \\
-\alpha_1 + 1 + {\alpha_0}^2 = 0, \\
\alpha_0\alpha_1 = 0, \\
\dfrac{{\alpha_1}^2}{3} = 0.
\end{cases}$$

From these algebraic equations we cannot get the value of α_1 because of made a counteraction. In fact for this type of nonlinear integral equation this method be failed.

Now, if we increase N and let $v_m(x) = x_m$ for $m = 0,1,2,\cdots$ then in view of Eq. (2.7) we have

$$u_0(x) = \sum_{m=0}^{\infty} \alpha_m \ x^m,$$
$$u_1(x) = -\sum_{m=0}^{\infty} \alpha_m \ x^m + x + \int_0^x (\sum_{m=0}^{\infty} \alpha_m \ t^m)^2 \, dt \ , \qquad (3.6)$$

Consequently, we have series solution of the form

$$y(x) = \sum_{m=0}^{\infty} \alpha_m \ x^m = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots = \tan(x),$$

which is the same as the exact solution.

Remark

We note that in this modified homotopy perturbation method, when algebraic equations cannot be solved and N is taken to be infinity, we obtain the series solution method for solving integral equations.

Theorem 1

The new modified homotopy perturbation [1 method] for solving integral equation is the series solution method when $N \longrightarrow \infty$ and $v_m(x) = x_m$ for $m = 0,1,2,\cdots$.

Proof. If $N \longrightarrow \infty$ and $v_m(x) = x_m$ for $m = 0,1,2,\cdots$, then in view of Eq. (2.7) we have

$$p^0: \qquad u_0(x) = \sum_{m=0}^{\infty} \alpha_m \ x^m,$$
$$p^1: \quad u_1(x) = -\sum_{m=0}^{\infty} \alpha_m \ x^m + g(x) + \int_0^x (\sum_{m=0}^{\infty} \alpha_m \ t^m)^r \ dt \qquad (3.7)$$

According to the modified homotopy perturbation method, we must consider $u_1(x) = 0$ then other components of $u(x)$, $u_2 = u_3 = \cdots = 0$, and the exact solution will be obtained as $y(x) = u_0(x)$. So, if we get $u_1(x) = 0$ then from (16) we have:

$$u_1(x) = 0 \ \rightarrow \sum_{m=0}^{\infty} \alpha_m \ x^m = g(x) + \int_0^x (\sum_{m=0}^{\infty} \alpha_m \ t^m)^r \ dt. \qquad (3.8)$$

In view of (3.8), it is easy to see this is well known the series solution method. Hence the proof is completed.

4 Discussion

In this note, by an example we have shown the new modified HPM presented by Ghorbani and Saberi-Nadjafi for solving nonlinear integral equations is not useful. In fact, when exact solution of integral equation is not appearing as part or a type of given function $g(x)$ this method is failed. In this case N is taken to be infinity, we obtain the series solution method for solving nonlinear integral equations. Another important result, if we select all selective function as xm then this method is equivalence the series solution method.

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